Final exam (practice)
UCLA: Math 31B, Spring 2017

Instructor: Noah White
Date:

- This exam has 8 questions, for a total of 80 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name:  

ID number:  

Discussion section:  

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Questions 1 and 2 are multiple choice. Once you are satisfied with your solutions, indicate your answers by marking the corresponding box in the table below.

*Please note! The following four pages will not be graded. You must indicate your answers here for them to be graded!*

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**Question 1.**

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**Question 2.**

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1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

(a) (2 points) Find $A$, $B$ and $C$ so that

$$\frac{4}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

A. $A = 1, B = 1, C = 2$
B. $A = 1, B = 1, C = 1$
C. $A = 1, B = -1, C = 2$
D. $A = 3, B = 0, C = 1$

(b) (2 points) Calculate the integral $\int_1^e \ln x \, dx$.

A. $e - 1$
B. $e$
C. $0$
D. $1$

(c) (2 points) Find an alternative expression for $\cos (\tan^{-1} x)$

A. $\frac{1}{\sqrt{1+x^2}}$
B. $\frac{x}{\sqrt{1-x^2}}$
C. $\frac{1}{\sqrt{1-x^2}}$
D. $\frac{\sqrt{1+x^2}}{x}$
(d) (2 points) The third Taylor polynomial of $\ln(x + 1)$ at 0 is
A. $x - x^2 + 2x^3$
B. $1 + x + x^2 + x^3$
C. $1 - x + 2x^2$
D. $x - \frac{1}{2}x^2 + \frac{1}{3}x^3$

(e) (2 points) Find the limit $\lim_{n \to \infty} \left( \frac{n}{n+1} \right)^n$.
A. $e^{-1}$
B. 1
C. 0
D. $e^2$

(f) (2 points) Calculate the improper integral $\int_1^{\infty} e^{-3x} \, dx$.
A. $(3e^3)^{-1}$.
B. $e^{-1}$.
C. $e$.
D. $3e$. 
2. In each of the following questions, analyse the integral, sequence or series below and determine whether it converges or diverges. No partial points will be given.

(a) (2 points)

\[ \int_{1}^{\infty} \frac{1}{x^2 \sqrt{\ln(x)}} \, dx \]

\( \checkmark \) Converges  
\( \bigcirc \) Diverges

(b) (2 points) The sequence \((a_n)\) where

\[ a_n = \frac{n \cos(\pi n)}{n + 1}. \]

\( \bigcirc \) Converges  
\( \checkmark \) Diverges

(c) (2 points) The sequence \((a_n)\) where

\[ a_n = (-1)^n \frac{1}{1 + \ln n}. \]

\( \checkmark \) Converges  
\( \bigcirc \) Diverges
(d) (2 points) The series
\[
\sum_{n=1}^{\infty} \frac{n\sqrt{n}}{n^4 + \ln n}
\]
- Converges
- Diverges

(e) (2 points) The series
\[
\sum_{n=3}^{\infty} \frac{e^{4n}}{(n-2)!}
\]
- Converges
- Diverges

(f) (2 points) The series \(\sum_{n=1}^{\infty} a_n\) where the partial sums are given by
\[
S_N = \frac{\ln n + 1}{\ln n}
\]
- Converges
- Diverges
3. (8 points) Calculate the following antiderivative.

\[ \int \frac{4x(x^2 - 2x + 4)}{x^4 - 16} \, dx \]

**Solution:** If

\[ \frac{4x(x^2 - 2x + 4)}{x^4 - 16} = \frac{A}{x - 2} + \frac{B}{x + 2} + \frac{Cx + D}{x^2 + 4}, \]

then

\[ 4x(x^2 - 2x + 4) = A(x + 2)(x^2 + 4) + B(x - 2)(x^2 + 4) + (Cx + D)(x - 2)(x + 2). \]

Let \( x = 2 \) to see \( 8 \cdot 4 = 4 \cdot 8 \cdot A \) and \( A = 1 \).

Let \( x = -2 \) to see \( -8 \cdot 12 = -4 \cdot 8 \cdot B \) and \( B = 3 \).

Let \( x = 0 \) to see \( 0 = 8A - 8B - 4D \) so \( D = 2(A - B) = -4 \).

From the hint, \( C = 0 \). So

\[ \int \frac{4x(x^2 - 2x + 4)}{x^4 - 16} \, dx = \int \left( \frac{1}{x - 2} + \frac{3}{x + 2} - \frac{4}{x^2 + 4} \right) \, dx \]

\[ = \ln |x - 2| + 3 \ln |x + 2| - 2 \arctan \left( \frac{x}{2} \right) + c. \]
4. (a) (2 points) Is there a function $f(x)$ such that the 4-th Taylor polynomial centered at 0 is given by

$$T_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}.$$ 

If so, give an example. No justification is required.

(b) (2 points) Is there a function $f(x)$ such that the 4-th Taylor polynomial centered at 0 is given by

$$T_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!}.$$ 

If so, give an example. No justification is required.

(c) (2 points) Is there a function $f(x)$ such that the 3-rd Taylor polynomial centered at 0 is given by

$$T_3(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}.$$ 

If so, give an example. No justification is required.

(d) (5 points) Let $f(x) = \cos x + \sin x$ and $T_n(x)$ be the $n$-th Taylor polynomial of $f(x)$ centered at 0. Find an $n$ such that

$$|T_n(2) - f(2)| \leq \frac{1}{10^{123}}.$$ 

Solution:

(a) Yes, $f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$.

(b) Yes, $f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!}$.

(c) No, since $T_3(x)$ cannot have degree 4.

(d) Note that $|f^{(n)}(x)| = |\cos x \pm \sin x| \leq 2$ for all $x$ and so

$$|T_n(2) - f(2)| \leq \frac{2 \cdot 2^{n+1}}{(n+1)!}.$$ 

We would like to pick an $n$ so that this fraction is less than $10^{-123}$. Notice that

$$\frac{2 \cdot 2^{n+1}}{(n+1)!} = \frac{2 \cdot 2 \cdot \ldots \cdot 2}{1 \cdot 2 \cdot \ldots \cdot (n+1)} \leq \frac{2^{n-18}}{20 \cdot 21 \cdot \ldots \cdot (n+1)}.$$ 

Since $2^{19} < 19!$. And this is less than

$$\frac{2^{n-18}}{20^{n-18}} = \frac{1}{10^{n-18}}.$$ 

In summary we have found that

$$|T_n(2) - f(2)| \leq \frac{2 \cdot 2^{n+1}}{(n+1)!} \leq \frac{1}{10^{n-18}}.$$ 

Thus we just need to make sure that $n - 18 \geq 123$. Taking $n = 141$ will do.
5. (a) (5 points) Use $u$-substitution(s) to calculate the following antiderivative.

$$\int \frac{2\ln(-\ln x)}{x\ln x} \, dx$$

(b) (4 points) Verify whether the following improper integral converges or diverges. If it converges, calculate what it converges to.

$$\int_{\frac{1}{2}}^{1} \frac{2\ln(-\ln x)}{x\ln x} \, dx$$

Solution:

(a) Let $u = \ln x$. Then $du = \frac{1}{x} \, dx$, so

$$\int \frac{2\ln(-\ln x)}{x\ln x} \, dx = \int \frac{2\ln(u)}{u} \, du.$$ 

Let $w = \ln(-u)$. Then $dw = \frac{1}{u} \, du$, so

$$\int \frac{2\ln(u)}{u} \, du = \int 2w \, dw = w^2 + c$$

and

$$\int \frac{2\ln(-\ln x)}{x\ln x} \, dx = \ln(-\ln x)^2 + c.$$ 

(b)

$$\int_{\frac{1}{2}}^{1} \frac{2\ln(-\ln x)}{x\ln x} \, dx = \lim_{S \to 1-} \int_{\frac{1}{2}}^{S} \frac{2\ln(-\ln x)}{x\ln x} \, dx$$

$$= \lim_{S \to 1-} \left[ \ln(-\ln x)^2 \right]_{\frac{1}{2}}^{S} = \lim_{S \to 1-} \left[ \ln(-\ln S)^2 \right] = \infty$$

The integral diverges.
6. (a) (4 points) Calculate the 4th Taylor polynomial centered at $x = 0$ for $\ln(3x + 4)$.

(b) (6 points) Calculate the Taylor series centered at 0 for $\ln(3x + 4)$.

**Solution:**

(a) Notice that

$$f^{(n)}(x) = (-1)^{n-1} \frac{3^n(n-1)!}{(3x + 4)^n}.$$  

So

$$f^{(n)}(0) = (-1)^{n-1} \left( \frac{3}{4} \right)^n (n-1)!.$$  

Thus

$$\frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n-1}}{n} \left( \frac{3}{4} \right)^n.$$  

Hence

$$T_4(x) = \ln 4 + \frac{3}{4} x - \frac{9}{32} x^2 + \frac{27}{192} x^3 - \frac{81}{1024} x^4.$$  

(b) Using the above

$$T(x) = \ln 4 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left( \frac{3}{4} \right)^n x^n.$$
7. (a) (4 points) Using the geometric series, obtain a power series with center $c = 0$ for the function

\[
\frac{1}{x + 1}.
\]

(b) (4 points) Using the properties of derivatives and integrals of power series, find a power series with center $c = 0$ for

\[
\ln(1 + x^2).
\]

Be sure to state the radius of convergence in both cases carefully, and to justify your choice of integration constant if necessary.

**Solution:**

(a) The geometric series says that

\[
\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n \quad \text{as long as } |x| < 1.
\]

Subbing in $-x$, we get that

\[
\frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n \quad \text{as long as } |-x| = |x| < 1.
\]

(b) Notice that

\[
\frac{d}{dx} \ln(1 + x^2) = \frac{2x}{1 + x^2}.
\]

From part a) we get that

\[
\frac{1}{1 + x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad \text{as long as } |x^2| < 1.
\]

which is the same as saying that $|x| < 1$. Now we simply multiply by $2x$ to get

\[
\frac{2x}{1 + x^2} = \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1} \quad \text{as long as } |x| < 1.
\]

To get the power series for $\ln(1 + x^2)$ we use the fact that we can integrate to get

\[
\ln(1 + x^2) = \int \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1} \, dx
\]

\[
= C + \sum_{n=0}^{\infty} \frac{(-1)^n 2x^{2n+2}}{2n + 2}
\]

\[
= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n + 1} x^{2(n+1)}
\]

For some unknown constant of integration. Using the fact that when $x = 0$, $\ln(1 + x^2) = 0$ we see that $C = 0$. Since integrating a power series does not change its interval of convergence, we get that whenever $|x| < 1$,

\[
\ln(1 + x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n + 1} x^{2(n+1)}.
\]
8. (a) (3 points) Does the series
\[
\sum_{n=0}^{\infty} \frac{1}{4n^2 - 1}
\]
converge? Justify your answer carefully.

(b) (7 points) Evaluate the series above. *Hint: use the definition of the series as a limit of partial sums. Calculate the partial sum by first using partial fractions.*

**Solution:**
(a) Let \(a_n = \frac{1}{4n^2 - 1}\) and \(b_n = n^2\). Then
\[
\lim_{n \to \infty} \frac{|a_n|}{b_n} = \frac{1}{4}
\]
and since \(\sum b_n\) converges (by the p-test), the limit comparison test shows that the series in the question also converges.

(b) First notice (by partial fractions) that
\[
\frac{1}{4n^2 - 1} = \frac{1}{2(2n - 1)} - \frac{1}{2(2n + 1)}.
\]

Now the definition of the infinite series says that
\[
\sum_{n=0}^{\infty} \frac{1}{4n^2 - 1} = \lim_{N \to \infty} S_N\quad \text{where}\quad S_N = \sum_{n=0}^{N} \frac{1}{4n^2 - 1}
\]

Now
\[
S_N = \sum_{n=0}^{N} \frac{1}{2(2n - 1)} - \frac{1}{2(2n + 1)}
\]
\[
= \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{6} + \frac{1}{6} - \frac{1}{10} + \ldots + \frac{1}{2(2N - 1)} - \frac{1}{2(2N + 1)}
\]
\[
= \frac{1}{2} - \frac{1}{2(2N + 1)}.
\]

where we see that we get lots of cancellation between neighbouring terms. Now we can simply take the limit
\[
\sum_{n=0}^{\infty} \frac{1}{4n^2 - 1} = \lim_{N \to \infty} S_N = \lim_{N \to \infty} \frac{1}{2} - \frac{1}{2(2N + 1)} = \frac{1}{2}.
\]
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