Note that the binomial coefficient $\binom{n}{k}$ counts the number of size-k subsets of a set of size n. For concreteness, let $[n] = \{1, 2, \ldots, n\}$; we'll look at subsets of $[n]$. A few of these could be proved algebraically from the formula.

1. $\binom{n}{k} = \binom{n}{n-k}$

   We have a bijection between subsets of size k and subsets of size $n - k$ given by matching up every set with its complement:

   $$\Phi : A \mapsto [n] - A$$

2. $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

   We can split the subsets of $[n]$ of size k into two parts: those that do not contain 1 (of which there are $\binom{n-1}{k}$) and those that do (of which there are $\binom{n-1}{k-1}$).

3. $\binom{n}{k-1} < \binom{n}{k}$ whenever $k \leq n/2$

   Construct a bipartite graph with all the subsets of size $k - 1$ on the left and all the subsets of size k on the right. Connect two vertices with an edge if one set is contained in the other.

   We count the number of edges in two different ways: every vertex on the left has degree $n - (k - 1)$, so there are $\binom{n}{k-1}(n - (k - 1))$ edges. And every vertex on the right has degree k, so there are $\binom{n}{k}(k)$ edges.

   So $\binom{n}{k-1}(n - (k - 1)) = \binom{n}{k}(k)$ which means

   $$\frac{\binom{n}{k}}{\binom{n}{k-1}} = \frac{n - (k - 1)}{k} > 1$$

   (We could also construct an explicit injection, but I wanted to emphasize double-counting here).

4. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

   The left-hand side counts the number of subsets of $[n]$ of any size, which is 2^n (every element is either in the set or not).

   This could also be done with the binomial theorem.

5. $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

   For the sum to be zero we need the positive (even) terms to cancel with the negative (odd) terms. So this is equivalent to saying the number of even size subsets is equal to the number of odd size subsets.

   One bijection is given by $\Phi : A \mapsto A \triangle \{1\}$ (just switch whether or not your subset contains 1).

   This could also be done with the binomial theorem.

6. $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

   Split $[2n]$ into two halves; then in order to pick n elements we have to pick k elements from the first half and *not* pick k elements from the second half.

7. $\sum_{n=0}^{m} \binom{n}{k} = \binom{m+1}{k+1}$

   The right-hand side counts the number of subsets of size $k+1$, while $\binom{n}{k}$ counts the number of such subsets whose largest element is $n+1$.