1. **Method-of-moments against MLE**. Suppose X_i are a random sample from $\mathrm{Unif}[-\theta, \theta]$ for θ a positive parameter. Note that the variance of a random variable with distribution $\mathrm{Unif}[a, b]$ is $(b-a)^2/12$.

   (a) Show that the method-of-moments estimator for θ is

   $$\sqrt{\frac{3}{n}\sum_{i=1}^{n} x_i^2}$$

   (b) Show that this estimator is biased. (Hint: Jensen's inequality[1]; computing the expectation directly is difficult).

   (c) Show that the MLE is $\max |x_i|$, which is also biased.

2. **Sufficient statistics**. Let X_i be a random sample from $\mathrm{Poisson}(\lambda)$ for a positive parameter λ. Consider the statistic $T = \sum x_i$. Show that T is sufficient in two different ways:

   (a) Directly, by computing the conditional distribution of X given T and showing that it does not depend on θ. (Hint: if A and B are independent $\mathrm{Poisson}(\lambda)$ and $\mathrm{Poisson}(\mu)$ respectively, then $A + B$ is $\mathrm{Poisson}(\lambda + \mu)$.

   (b) Using the Fischer-Neyman factorization theorem.

[1] Jensen's inequality says that for a random variable X with expectation and a convex function f,

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)].$$

Moreover, if f is strictly convex, then equality holds if and only if X is almost surely constant.

In our problem, this means

$$\mathbb{E}\left[\sqrt{Y}\right] \leq \sqrt{\mathbb{E}[Y]}$$

with equality if and only if Y is almost surely constant.

(1)

(a)

Let's find the moments of X_i, using the known moments of the uniform distribution:

$$\mathbb{E}[X_i] = \frac{\theta + (-\theta)}{2} = 0$$

Since the first moment is zero, we get no information from the first moment. So we need to continue to the second moment:

$$\mathbb{E}[X_i] = \frac{(\theta - -\theta)^2}{12} = \frac{\theta^2}{3}$$

So the method of moments gives us

$$\frac{\hat{\theta}^2}{3} = \frac{1}{n} \sum x_i^2$$

which we can solve to get the desired estimator.

(b)

To check bias, we compute the expectation:

$$\begin{aligned}
\mathbb{E}[\hat{\theta}] &= \mathbb{E}\left[\sqrt{\frac{3}{n} \sum X_i^2}\right] \\
&\overset{\text{(Jensen)}}{<} \sqrt{\mathbb{E}\left[\frac{3}{n} \sum X_i^2\right]} \\
&= \sqrt{\frac{3}{n} \sum \mathbb{E}[X_i^2]} \\
&= \sqrt{\frac{3}{n} \cdot n \frac{\theta^2}{3}} \\
&= \theta
\end{aligned}$$

Since we have an inequality, that means that it's a biased estimator.

(2)

(a)

First, note that the distribution of T is Poisson with parameter $n\theta$. Let $(x_1, \ldots, x_n)$ be a sequence of nonnegative integers. Then

$$\begin{aligned}
\mathbb{P}(X_1 = x_1, \ldots, X_n = x_n | T = t) &= \frac{\mathbb{P}(X_1 = x_1, \ldots, X_n = x_n, T = t)}{\mathbb{P}(T = t)} \\
&= \frac{\mathbb{P}(X_1 = x_1, \ldots, X_n = x_n)}{\mathbb{P}(T = t)} \mathbb{1}_{\sum x_i = t} \\
&= \left(\prod_{i=1}^{n} \frac{e^{-\theta} \theta^{x_i}}{x_i!} \right) \cdot \frac{t!}{e^{-n\theta}(n\theta)^t} \mathbb{1}_{\sum x_i = t} \\
&= \frac{(e^{-\theta})^n \theta^{\sum x_i}}{e^{-n\theta}\theta^t} \cdot \frac{t!}{x_1! x_2! \cdots x_n!} \cdot \frac{1}{n^t} \cdot \mathbb{1}_{\sum x_i = t} \\
&= \binom{t}{x_1, x_2, \ldots, x_n} \cdot \frac{1}{n^t} \cdot \mathbb{1}_{\sum x_i = t}
\end{aligned}$$

which does not depend on θ, as required. Note that we could assume $\sum x_i = t$, since if this doesn't happen then the $\mathbb{1}_{\sum x_i = t}$ term will eliminate things.

(b)

Let $(x_1, \ldots, x_n)$ be nonnegative integers. We can write

$$\begin{aligned}
\mathbb{P}_\theta((x_1, \ldots, x_n)) &= \mathbb{P}_\theta(X_1 = x_1, \ldots, X_n = x_n) \\
&= \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{x_i}}{x_i!} \\
&= \frac{e^{-n\theta}\theta^{\sum x_i}}{x_1! \cdots x_n!} \\
&= f_\theta(T(x)) \cdot \frac{1}{x_1 \cdots x_n!}
\end{aligned}$$

where $f_\theta(t) = e^{-n\theta}\theta^t$. So the distribution decomposes into a function of t and θ times a function of x, which exactly means the statistic is sufficient by the decomposition theorem.