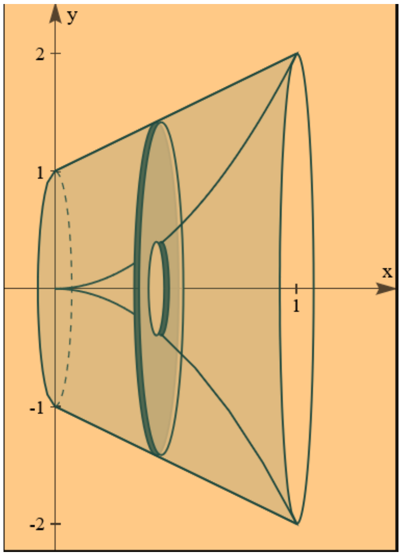
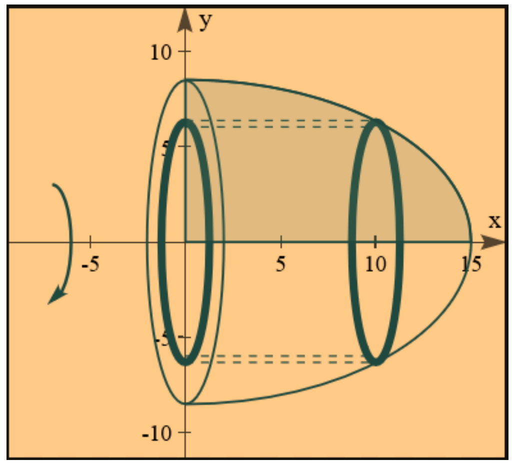
Solids of revolution [6.3, 6.4]





We have two methods of finding the volume of a solid of revolution - by seeing it as having cross-sections in the shape of a washer (an annulus with a very small *height*) or by seeing it as having cross-sections in the shape of a cylindrical shell (a cylinder with a very small *thickness*). Note that both of these pictures are for rotation around a horizontal axis; if we want to rotate around a vertical axis, we just need to flip x and y.

Washers

Take a look at the first picture above. That solid has vertical cross-sections that look like very thin washers. Each washer has a volume of

$$\text{(volume of washer)} = \text{(thickness)} \cdot \underbrace{\left[\pi(\text{outer radius})^2 - \pi(\text{inner radius})^2\right]}_{\text{surface area}}$$

which means that if we integrate through x, we will get the volume of the whole solid as

$$\text{(volume of solid)} = \int_a^b \left[\pi(\text{outer radius at } x)^2 - \pi(\text{inner radius at } x)^2\right] \, dx$$

Here a and b are the endpoints of our region. If we let the outer bound be $f(x)$ and the inner bound be $g(x)$, and rotate about the line $y = k$, then we get

$$\text{(volume of solid)} = \pi \int_a^b (f(x) - k)^2 - \pi(g(x) - k)^2 \, dx$$

Note that $f(x) - k$ is the *distance* between $f(x)$ and k, and is thus the outer radius. The same is true of $g(x) - k$. In the case that the inner radius is exactly zero (in that we are rotating about one edge of our region) this is called the disk method.

Washers

Take a look at the second picture above. That solid has radial cross-sections that look like very thin cylinders. Each washer has a volume of approximately

$$\text{(volume of washer)} \approx \text{(thickness)} \cdot 2\pi\text{(radius)} \cdot \text{(height)}.$$

which means that if we integrate through y, we will get the volume of the whole solid as

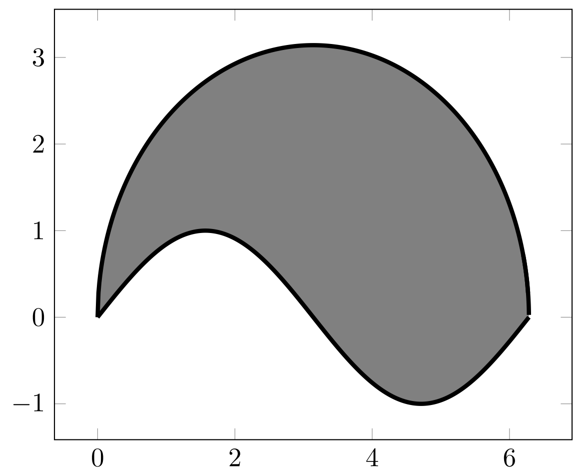
$$\text{(volume of solid)} = \int_c^d 2\pi\text{(radius)} \cdot \text{(height)}\ dy$$

Here c and d are the endpoints of our region. Suppose our left bound is given by $x = r(y)$ and our right bound by $x = s(y)$, and that we rotate about the line $x = \ell$. For a particular shell at y, its radius is $y - \ell$, and its height is $s(y) - r(y)$. So we get that the volume is

$$\text{(volume of solid)} = \int_c^d 2\pi(y - k) \cdot (s(y) - r(y))\ dy$$

For each of the following problems, set up but do not evaluate an integral or integrals.

1. Consider the area between the graphs of $y = \sin(x)$ and a circle[1] of radius π centered at $(\pi, 0)$:



   Find the volume of the shape generated by rotating this area around the line $y = 5$. Then find the volume of the shape generated by rotating this area around the line $x = -1$.

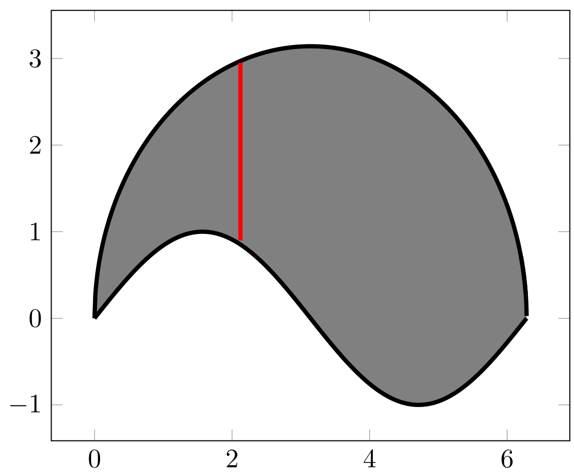
2. Consider the area in the xy-plane defined by $|x| + |y| \leq 1$, which is a square with vertices at $(\pm 1, 0)$ and $(0, \pm 1)$. Find the volume obtained by rotating this shape about the line $x = 2$. Do this in two different ways.

[1] The equation of this circle is

$$(x - \pi)^2 + y^2 = \pi^2$$

Solutions

In order to rotate around the line $y = 5$, use washers to set up the integral:



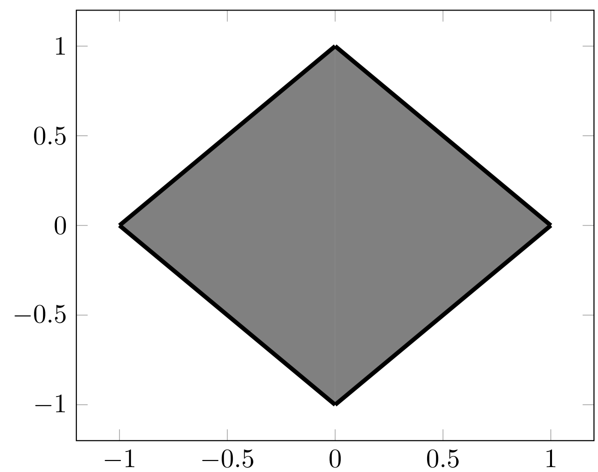
$$\begin{aligned}
\text{(volume)} &= \int_0^{2\pi} \text{(area of washer at } x\text{)}\ dx \\
&= \int_0^{2\pi} \pi\left[(\text{outer radius at } x)^2 - (\text{inner radius at } x)^2\right] \\
&= \int_0^{2\pi} \pi\left[(5-\sin x)^2 - (5-\sqrt{\pi^2-(x-\pi)^2})^2\right]\ dx
\end{aligned}$$

- Note that if we had tried to use shells instead, we would have to set up our integration using horizontal instead of vertical strips, which would be much harder with the bounds we are given. In fact, we'd have to set up *four* different integrals instead of one.

To rotate about the line $x = -1$, we should use shells for the same reason; it's easier to set up an integral parallel to the axis of rotation rather then perpendicular. We get

$$\begin{aligned}
\text{(volume)} &= \int_0^{2\pi} \text{(area of cylinder at } x\text{)}\ dx \\
&= \int_0^{2\pi} 2\pi \cdot \text{(radius of cylinder)} \cdot \text{(height of cylinder)}\ dx \\
&= \int_0^{2\pi} 2\pi \cdot (x+1) \cdot \left(\sqrt{\pi^2-(x-\pi)^2} - \sin x\right)\ dx
\end{aligned}$$

- Note that the radius of the cylinder is the distance between the boundary and the axis of rotation; in this case, it's the distance between x and -1, namely $x+1$.



First, let's do this integral by the method of washers. That means we need to set up our slices to be *perpendicular* to the axis of rotation, so we're looking horizontally. So we get

$$\begin{aligned}
\text{(volume)} &= \int_{-1}^{1} \text{(area of washer at } y) \; dy \\
&= \int_{-1}^{1} \pi \left[(\text{outer radius at } y)^2 - (\text{inner radius at } y)^2 \right] \; dy \\
&= \int_{-1}^{0} \pi \left[(y+3)^2 - (-y+1)^2 \right] \; dy \\
&\quad + \int_{0}^{1} \pi \left[(-y+3)^2 - (y+1)^2 \right] \; dy
\end{aligned}$$

- We split the integral in two because these functions are defined piecewise: the equations for the inner and outer radius are different when $y < 0$ and $y > 0$.
- In this case, the two integrals are equal by symmetry; the top half of the resulting solid has the same shape as the bottom half. So it's enough to take one of the integrals and double it.

Now let's use cylindrical shells. That means our slices should be *parallel* to the axis of rotation, and we get

$$\begin{aligned}
(\text{volume}) &= \int_{-1}^{1} (\text{area of cylinder at } x)\ dx \\
&= \int_{-1}^{1} 2\pi \cdot (\text{radius of cylinder}) \cdot (\text{height of cylinder})\ dx \\
&= \int_{-1}^{0} 2\pi \cdot (2 - x) \cdot (2x + 2)\ dx \\
&\quad + \int_{0}^{1} 2\pi \cdot (2 - x) \cdot (2x - 2)\ dx
\end{aligned}$$

- Again, we had to use two integrals since the bounds are different for different parts of the solid.
- But here the two integrals are not the same; the inner part of the solid has a different volume from the outer part.