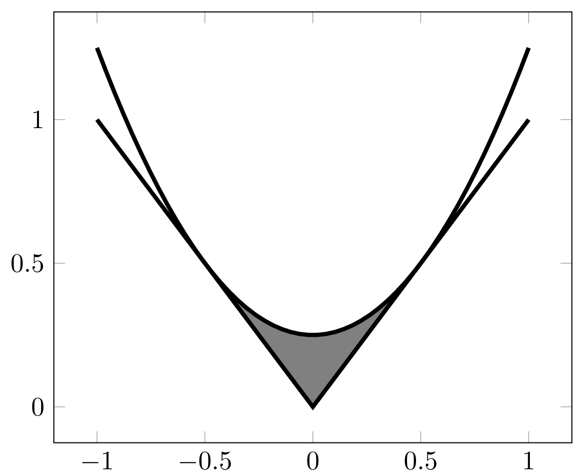
1. Compute the following integrals:

$$\int_0^1 3x\sin(x^2-5)\ dx \qquad \int (x-9)^{100}(x-8)\ dx$$

$$\int \sec x \cdot \sec x \cdot \tan x\ dx \qquad \int \frac{x^7+5}{x^6}\ dx$$

(For the lower left one: there are two plausible substitutions which both will work. Do they get you the same answer?)

2. Find the area between the graphs of $y = x^2 + 1/4$ and $y = |x|$. (The intersection points are at $x = \pm 1/2$).



3. Show that the volume of a cone with height 10 and basal radius 5 is $\frac{250}{3}\pi$ (that is, 1/3 of the volume of the cylinder containing it). Hint: what is the area of a vertical cross-section at a height of h?

Solutions

(1)

For the first integral, the substitution $u = x^2 - 5$ suggests itself; then we get

$$\begin{aligned}\int_0^1 3x\sin(x^2-5)\ dx &= \frac{3}{2}\int_0^1 \underbrace{2x}_{u'}\sin(\underbrace{x^2-5}_{u})\ dx\\ &= \frac{3}{2}\int_{-5}^{-4}\sin u\ du\\ &= \frac{3}{2}\left[-\cos u\Big|_{-4}^{-5}\right]\\ &= \frac{3}{2}\left[\cos(-5)-\cos(-4)\right]\\ &= \frac{3}{2}\left[\cos(5)-\cos(4)\right]\approx 1.4060\end{aligned}$$

Don't forget to change the bounds of integration when we do a substitution! In this case the bounds went from 0 and 1 to -5 and -4.

For the second, we'd like to let $u = x - 9$. In that case, the $x - 8$ term becomes $u + 1$, so we'll get

$$\begin{aligned}\int (x-9)^{100}(x-8)\ dx &= \int u^{100}(u+1)\ du\\ &= \int u^{101}+u^{100}\ du\\ &= \frac{u^{102}}{102}+\frac{u^{101}}{101}+C\\ &= \frac{(x-9)^{102}}{102}+\frac{(x-9)^{101}}{101}+C\end{aligned}$$

For the third, we can try the substitution $u = \sec x$:

$$\begin{aligned}\int \underbrace{\sec x}_{u}\cdot\underbrace{\sec x\cdot\tan x}_{u'}\ dx &= \int u\ du\\ &= \frac{u^2}{2}+C\\ &= \frac{\sec^2 x}{2}+C\end{aligned}$$

Or we could use $v = \tan x$:

$$\int \underbrace{\sec x \cdot \sec x}_{v'} \cdot \underbrace{\tan x}_{v} \; dx = \int v \; dv$$
$$= \frac{v^2}{2} + D$$
$$= \frac{\tan^2 x}{2} + D$$

Of course, since $\sec^2 x = 1 + \tan^2 x$, these are the same (up to constants).

The last one is a red herring; you probably don't want to use a substitution at it's just $\int x + 5x^{-6} \; dx$ which the power rule tells us is $x^2/2 - x^{-5} + C$.

2

We set up the integral as follows: the area between $y = x^2 + 1/4$ and $y = |x|$ is

$$\int_{-1/2}^{1/2} (x^2 + 1/4) - |x| \; dx$$

We need to write this integral in piecewise form: it's

$$\int_{-1/2}^{0} x^2 + x + 1/4 \; dx + \int_{0}^{1/2} x^2 - x + 1/4 \; dx$$

but because of the symmetry of the problem, it's enough to find the area of the right half and double it. So we get

$$\begin{aligned}(\text{area}) &= 2 \cdot (\text{area of right half}) \\ &= 2 \int_0^{1/2} x^2 - x - 1/4 \; dx \\ &= 2 \left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4} \Big|_0^{1/2} \right] \\ &= 2 \left[\frac{1}{24} - \frac{1}{8} + \frac{1}{8} \right] \\ &= \frac{1}{12}\end{aligned}$$