Integration

1. Find an antiderivative to the function

$$h(x) = 6x^2 + 2\sin x$$

   Use that antiderivative, and the FTC, to evaluate

$$\int_0^{\pi} h(x)\ dx.$$

   Now find another antiderivative, and use the FTC with your new antiderivative. Do you get the same answer?

2. A train begins its journey at a distance of k from the station, moving away from it. The velocity of a train at time t is given by

$$v(t) = 12 - 3t^2$$

   At time 4, the train has returned to the station, after which it stops. Find an equation to describe the position of the train (in terms of its distance from the station) at time t. At what time is the train furthest away from the station, and how far from the station is it at that point? How far from the station was the train when it started? What is the total (not net) distance traveled by the train between times 0 and 4?

3. The linearity rules tell us that we can distribute integration over summation as well as pull constants through the integral sign; that is, $\int f + g = \int f + \int g$ and $\int cf = c \int f$. The same does not hold for multiplication: Show that

$$\int x \cdot x\ dx \neq \left(\int x\ dx\right) \cdot \left(\int x\ dx\right)$$

4. Why can't we use the Power Rule to evaluate

$$\int x^{-1}\ dx\ ?$$

Solutions

1

Using the power rule and the antiderivatives of trig functions we come up with the antiderivative

$$H_1(x) = 2x^3 - 2\cos x$$

The FTC tells us that

$$\begin{aligned} \int_0^\pi h(x)\ dx &= H_1(\pi) - H_1(0) \\ &= (2\pi^3 - 2\cos\pi) - (2\cdot 0^3 - 2\cos 0) \\ &= (2\pi^3 + 2) - (0 - 2) \\ &= 2\pi^3 + 4 \approx 66.0126 \end{aligned}$$

Any other antiderivative is of the form

$$H_2(x) = H_1(x) + C$$

so if we use that we get

$$\begin{aligned} \int_0^\pi h(x)\ dx &= H_2(\pi) - H_2(0) \\ &= H_1(\pi) + C - (H_1(0) + C) \\ &= H_1(\pi) - H_1(0) \end{aligned}$$

as before.

2

Since velocity is the derivative of position, we have that the position must be given by an antiderivative of $v(t)$; that means it must look like $s(t) = 12t - t^3 + C$ for some unknown constant C.

Since we know that $s(4) = 0$, we get $0 = 12\cdot 4 - 4^3 + C$, so $C = 64 - 48 = 16$. Thus the position of the train at time t is $12t - t^3 + 16$.

To find the maximum distance, we need to check the critical points of s. There's one at each endpoint (0 and 4); and $s'(t) = v(t) = 0$ when $t = 2$ (or $t = -2$ but that's not in our interval). So we check these points:

$$s(0) = 12\cdot 0 - 0^3 + 16 = 16 s(2) \quad = 12\cdot 2 - 2^3 + 16 = 32 s(4) = 12\cdot 4 - 4^3 + 16 = 0$$

So the maximum distance is 32, and it happens at time 2.

We just computed where the train starts; it is at $s(0) = 16$.

To find the total distance covered, we need to integrate $|v|$, which is a piecewise function:

$$|v(t)| = \begin{cases} v(t) & v(t) \geq 0 \\ -v(t) & v(t) < 0 \end{cases} = \begin{cases} 12 - 3t^2 & v(t) \leq 2 \\ 3t^2 - 12 & v(t) > 2 \end{cases}$$

We integrate a piecewise function by looking at each piece separately:

$$\begin{aligned}
\int_0^4 |v(t)|\ dt &= \int_0^2 12 - 3t^2\ dt + \int_2^4 3t^2 - 12\ dt \\
&= (12t - t^3)\Big|_0^2 + (t^3 - 12t)\Big|_2^4 \\
&= (12 \cdot 2 - 2^3) - (12 \cdot 0 - 0^3) + (4^3 - 12 \cdot 4) - (2^3 - 12 \cdot 2) \\
&= 24 - 8 - 0 + 0 + 64 - 48 - 8 + 24 \\
&= 48
\end{aligned}$$

(We could just do this by seeing that the train goes from 16 to 32 and then back to 0. But it's nice to demonstrate how to integrate piecewise functions).

3

$\int x = x^2/2 + C$, and $\int x^2 = x^3/3 + D$. This means

$$\left(\int x\right)^2 = (x^2/2 + C_1)(x^2/2 + C_2)$$

which is a degree 4 polynomial that is not equal to $x^3/3$. This is a very common mistake made by calculus students.