Integration

1. Use a left Riemann sum with three equal intervals to estimate

$$\int_0^6 x^3 \, dx.$$

Now do one with six intervals. Are your answers overestimates or underestimates?

2. Find

$$\int_0^1 \sqrt{1-x^2} \, dx$$

without doing any computations. (Hint: What is the shape of this graph?)

3. Find

$$\int_{-\pi}^{\pi} \sin\theta \, d\theta$$

without doing any computations. (Hint: What happens to the integral when the function is negative?)

4. Define the function

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}.$$

Compute the left and right Riemann sums of

$$\int_0^1 f(x) \, dx$$

for N equal-length intervals. What happens as $N \to \infty$?

Solutions

1

Breaking the interval $[0,6]$ into three equal intervals means that we want to evaluate at $x=0$, $x=2$, $x=4$. So we get that the left-hand sum is

$$0^3 \cdot (2-0) + \underbrace{2^3}_{\text{height}} \cdot \underbrace{(4-2)}_{\text{width}} + 4^3 \cdot (6-4)$$
$$= 0 \cdot 2 + 8 \cdot 2 + 64 \cdot 2 = 0 + 16 + 128 = 144$$

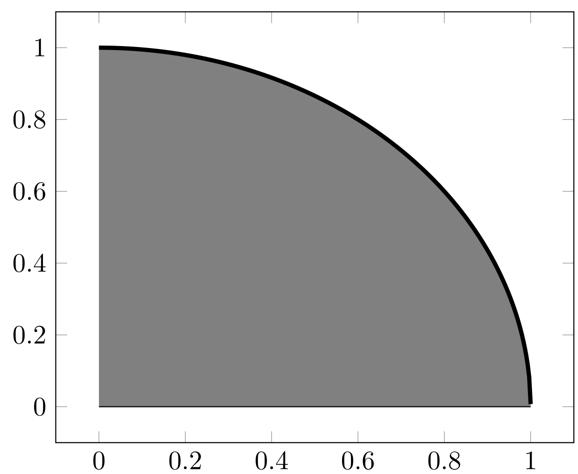
If we have six intervals, that means that they start at 0, 1, 2, 3, 4, and 5. So we get (since all the boxes have width 1):

$$1 \cdot (0^3 + 1^3 + 2^3 + 3^3 + 4^3 + 5^3) = 1 \cdot (0 + 1 + 8 + 27 + 64 + 125) = 225$$

Since we are taking *left* Riemann sums of an *increasing* function, all of the boxes will be below the actual graph (draw a picture). So these must both be underestimates (the actual answer is 324).

2

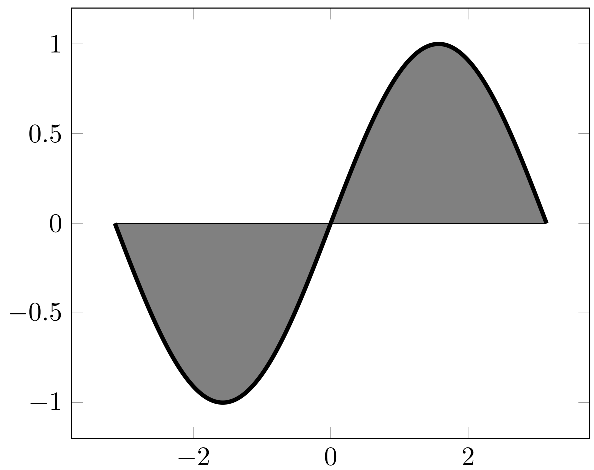
The area in question is exactly a quarter-circle of radius 1:



and so it has an area of $\pi/4$.

3

We are integrating an odd function on an interval symmetric about zero:



so the negative area on the left cancels out the positive area on the right, meaning that the total area is zero.

4

If we do a left Riemann sum, then all of the boxes will be of height zero except the first, which will be of height one. So the total area will be

$$1 \cdot \frac{1}{N} + 0 + \cdots + 0 = 1/N.$$

If we do a right Riemann sum, all the boxes will have height zero. So the total area will be zero. As $N \to \infty$, both kinds of Riemann sum will converge to the true integral, 0.