Review

1. Suppose we have some differentiable function f obeying the following:

x	$f(x)$	$f'(x)$
1	3	-2
2	3	-3
3	5	-2
4	4	2
5	4	3
6	3	2

   Estimate $f(f(2.99))$ by using a linear approximation for $f \circ f$ near 3.

2. You are chasing your friend who has just stolen your copy of Rogawski. Both of you run at exactly 2 m/s. Trying to evade you, your friend runs around a corner into an alley which is a right angle to the sidewalk you are currently on. At time $t = 0$, your friend is 10m away from the alley and you are 24m away. Let $x(t)$ be the distance between the two of you at time t.

   (a) How quickly is the distance between you changing at time $t = 8$?

   (b) What is the closest that you get to your friend, and at what time does this happen?

   (c) At the time in part (b), how quickly is the distance between you changing? (Hint: you should not need any computations).

   (d) Sketch the graph of $x(t)$ for t in the range $[0, 15]$. (Don't worry about concavity). For what values of t is $x(t)$ continuous? Differentiable?

   (e) Compare the values of $x(5)$ and $x(12)$. What theorem tells us about a particular value of x' in the interval $(5, 12)$? Check the hypotheses of the theorem and show that the conclusion is true.

3. Consider the function $y = \arcsin x$. Take the sin of both sides and use implicit differentiation to show that

$$y' = \frac{1}{\sqrt{1 - x^2}}$$

Solutions

1

First let's note

$$f(f(3)) = f(5) = 4$$
$$(f \circ f)'(3) = f'(f(3)) \cdot f'(3) = f'(5) \cdot f'(3) = 3 \cdot -2 = -6.$$

(We use the chain rule for the second one). So the linearization is

$$L(x) = f(f(3)) + (f \circ f)'(3)(x - 3) = 4 - 6(x - 3).$$

If we plug in $x = 3.01$ that gives

$$L(3.01) = 4 - 6(2.99 - 3) = 4 + 60.01 = 4.06$$

2(a)

When $t < 5$ or $t > 12$, both you an your friend are on the same side of the corner and so the distance is exactly 14. But in the interval $[5, 12]$, we have that you are $24 - 2t$ away from the corner and your friend is $2t - 10$ away from the corner, so the distance is

$$x(t) = \begin{cases} \sqrt{(2t-10)^2 + (24-2t)^2} & 5 \leq t \leq 12 \\ 14 & \text{otherwise} \end{cases}$$

In particular, $x^2 = (2t - 10)^2 + (24 - 2t)^2$ (for $t \in [5, 12]$), so differentiating gives

$$2x\frac{dx}{dt} = 4(2t - 10) - 4(24 - 2t) = 8t - 68$$

which means that when $t = 8$,

$$\frac{dx}{dt} = \frac{8t - 68}{2\sqrt{(2t-10)^2 + (24-2t)^2}} = \frac{64 - 68}{2\sqrt{6^2 + 8^2}} = -\frac{2}{5}$$

2(b)

We want to set dx/dt to be zero: this happens when

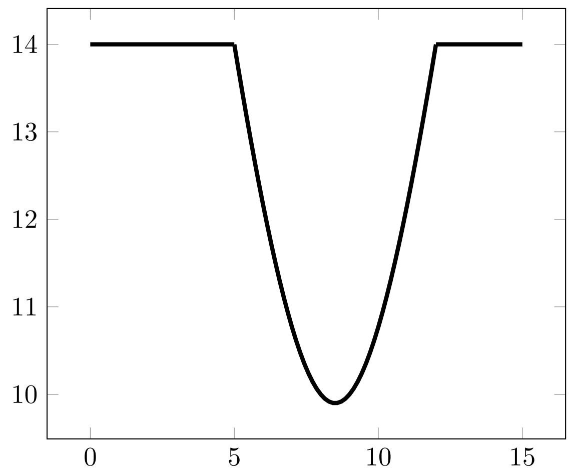
$$\frac{8t - 68}{2\sqrt{(2t-10)^2 + (24-2t)^2}} = 0$$

so $8t - 68 = 0$, or $t = 17/2$. At this time, you and your friend are both 7m from the corner, meaning the minimum is $7\sqrt{2} \approx 9.9$m

2(c)

At a minimum, the derivative must be zero.

2(d)



Note that x is continuous at 5 and 12 (since the left-hand and right-hand limits work out) but it is not diff. at either (since the limits of the derivative do not agree).

2(e)

The mean value theorem / Rolle's theorem guarantees us that, since $f(5) = f(12)$ and f is diff. on $(5, 12)$ and cts. at 5 and 12, there must be some $c \in (5, 12)$ such that $f'(c) = 0$. And there is; we found it in part (c).

3

If $y = \arcsin x$, then $\sin y = x$. Differentiating w/r/t x we get

$$\cos y \cdot \frac{dy}{dx} = 1$$

That means

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1 - \sin^2(\arcsin x)}} = \frac{1}{\sqrt{1 - x^2}}$$