1. Find and classify all the local minima and maxima of the following functions:

   (a) $f(x) = x^4/4 - x^3/3 - x^2/2 + x + 1$. (Hint: 1 is a root of g').

   (b) $\mu(x) = x + \sin x$

   (c) $\aleph(x) = \begin{cases} \sin(x) & x \leq 0 \\ x - x^3 & x > 0 \end{cases}$

2. Find the closest point on the parabola $y = x^2$ to the point $(3, 3)$. (Hint: $4x^3 - 10x - 6 = (x+1)(4x^2 - 4x - 6)$). Is there another (local) minimum?

3. (For if you have extra time). Show that a polynomial of *even* degree always has at least one critical point, but that the same is not true for polynomials of *odd* degree.

Solutions

1(a)

Let's take the derivative:

$$f'(x) = x^3 - x^2 - x + 1.$$

If we take the hint and factor out $(x-1)$ then we get

$$f'(x) = x^2(x-1) - 1(x-1) = (x-1)^2(x+1)$$

So our critical points are ± 1. Let's observe the sign of f': it is negative for $x < -1$, while it is positive for $-1 < x < 1$ and $1 < x$. So there is a local min at $x = -1$, where $f(x) = 1/12$, and no local extremum at $x = 1$.

1(b)

Taking the derivative we get

$$\mu'(x) = 1 + \cos x$$

which is zero infinitely often. But $\mu'(x) \geq 0$ everywhere, which means by the first derivative test there are no local extrema.

1(c)

First, observe that the function $\aleph$ is actually continuous at zero, since the left and right hand limits agree there. So we can actually apply our techniques to this function. The derivative is

$$\aleph'(x) = \begin{cases} \cos(x) & x < 0 \\ 1 - 3x^2 & x > 0 \end{cases}$$

and since $\lim_{x\to 0^+} \aleph'(x) = \lim_{x\to 0^-} \aleph'(x) = 1$, we can fill in that $\aleph'(0) = 1$. So this is not a critical point. The critical points are those where $\cos(x) = 0$ (integer + 1/2 multiples of π) and the positive root of $1 - 3x^2$, which is $1/\sqrt{3}$.

We know the local mins and maxes of $\sin(x)$, so we just need to check the positive critical point: and $\aleph''(1/\sqrt{3}) = -6(1/\sqrt{3}) < 0$, so this is a maximum.

2

Every point on the parabola $y = x^2$ is of the form (x, x^2) for some x. The distance between this point and $(3,3)$ is, as a function of x,

$$r(x) = \sqrt{(x-3)^2 + (x^2-3)^2}$$

We can save ourselves a little bit of work by noticing that minimizing r is the same thing as minimizing r^2 - as r is nonnegative, it and r^2 will have minima at the same points. So we write

$$\begin{aligned} r^2(x) &= (x-3)^2 + (x^2-3)^2 \\ &= x^2 - 6x + 9 + x^4 - 6x^2 + 9 \\ &= x^4 - 5x^2 - 6x + 18 \end{aligned}$$

Taking the derivative gives

$$(r^2)'(x) = 4x^3 - 10x^2 - 6 = (x+1)(4x^2 - 4x - 6)$$

which has as its zeros -1 and

$$\frac{4 \pm \sqrt{16+96}}{8} = \frac{1 \pm \sqrt{7}}{2}$$

From looking at the picture we can tell we want the positive root; we could check this with the first or second derivative tests. So our closest point is

$$\left(\frac{1+\sqrt{7}}{2}, \left[\frac{1+\sqrt{7}}{2} \right]^2 \right) = \left(\frac{1+\sqrt{7}}{2}, \frac{4+\sqrt{7}}{2} \right) \approx (1.8229, 3.3229)$$

3

If we take the derivative of an even degree polynomial, we get an odd degree polynomial, and odd degree polynomials always have real roots[1]. The same is not true for odd degree polynomials: for n odd, look at

$$x^n + x$$

This has derivative $nx^{n-1} + 1$, which is always positive since $n-1$ is even. So this function has no critical points.

[1] We can prove this by seeing that the limits to $\pm\infty$ have opposite signs and then using the IVT.