1. Find the critical points and extreme values of the given functions on the specified domains:

   (a) $f(x) = x^4 - 8x^2 + 8$ on $[-3, 5]$.

   (b) $h(x) = |\sin x + 1/2|$ on $[0, 2\pi]$.

   (c) $g(x) = \cos(1/x)$ on $[-1, 1]$, where we define $g(0) = 0$.

2. Suppose we have a twice-differentiable function f where $f(0) = f(1) = f(2) = 0$. Show that there must exist some $x \in [0, 2]$ such that $f''(x) = 0$. (Hint: Use a theorem three times).

3. (Forward looking). Suppose we have some many-times differentiable function f and we want to approximate it near zero. We know that the *linear* approximation is

   $$f(0) + f'(0)x.$$

   This is a linear function that agrees with f in its zeroth and first derivatives at 0. If we instead try a *quadratic* approximation, then because of the extra term we can make a function that agrees with f in its zeroth, first, and second derivatives at 0. Determine what this approximation should be (in terms of $f(0)$, $f'(0)$, and $f''(0)$). Now try it with a cubic - this time you should have agreement of the first three derivatives. What would an approximating polynomial of degree n look like? What should happen as n gets big?

Solutions

1(a)

Take the derivative to get

$$f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x-2)(x+2)$$

This has roots 0 and ± 2 in the given interval. So we have to check these three points in addition to the endpoints:

$$\begin{aligned} f(-3) &= (-3)^4 - 8(-3)^2 + 8 = 17 \\ f(-2) &= (-2)^4 - 8(-2)^2 + 8 = -8 \\ f(0) &= (0)^4 - 8(0)^2 + 8 = 8 \\ f(2) &= (2)^4 - 8(2)^2 + 8 = -8 \\ f(5) &= (5)^4 - 8(5)^2 + 8 = 433 \end{aligned}$$

We can see that the maximum is $f(5) = 433$ and the minimum is tied between $f(2) = -8$ and $f(-2) = -8$. (Note that since f is an even function we know that $f(2) = f(-2)$).

1(b)

Recall that the derivative of $|x|$ is

$$(\text{abs})'(x) = \begin{cases} -1 & x < 0 \\ \text{undef.} & x = 0 \\ 1 & x > 0 \end{cases}$$

So, using the chain rule we get

$$h'(x) = \begin{cases} -\cos x & \sin x + 1/2 < 0 \\ \text{undef.} & \sin x + 1/2 = 0 \\ \cos x & \sin x + 1/2 > 0 \end{cases}$$

meaning that the critical points are where $\cos x = 0$ or $\sin x + 1/2 = 0$. These are the points $\pi/2$, $3\pi/2$ (from the first condition) and $7\pi/6$ and $11\pi/6$ (from the second condition). So we have six points to check:

$$\begin{aligned} h(0) &= |\sin 0 + 1/2| = 1/2 \\ h(\pi/2) &= |\sin(\pi/2) + 1/2| = 3/2 \\ h(7\pi/6) &= |\sin(7\pi/6) + 1/2| = 0 \\ h(3\pi/2) &= |\sin(3\pi/2) + 1/2| = 1/2 \\ h(11\pi/6) &= |\sin(11\pi/6) + 1/2| = 0 \\ h(2\pi) &= |\sin 2\pi + 1/2| = 1/2 \end{aligned}$$

So the maximum occurs at $h(\pi/2) = 3/2$, and the minimum twice, at $h(7\pi/6) + h(11\pi/6) = 0$. (Note that the second one has to be the minimum since h is nonnegative).

1(c)

We can take the derivative to find

$$g'(x) = \left(-\frac{1}{x^2}\right) \cdot (-\sin(1/x)) \qquad (x \neq 0)$$

Setting this equal to zero, we see that the first factor is never zero. If the second factor is zero, it means $\sin(1/x) = 0$, so $1/x = n\pi$ for some integer n. That means the critical points are $1/(n\pi)$ for nonzero[1] integers n as well as the point $x = 0$ where the derivative does not exist.

Plugging in the (infinitely many) critical values we get

$$\begin{aligned} g(1/(n\pi)) = \cos(n\pi) &= \begin{cases} 1 & n \text{ even} \\ -2 & n \text{ odd} \end{cases} \\ g(0) &= 0. \end{aligned}$$

So the minimum is -1 which is attained at all of the points $x = 1/(n\pi)$ where n is odd; and the maximum is 1 which is attained at all of the points $x = 1/(n\pi)$ where n is even and nonzero.

2

The MVT (or Rolle's theorem) tells us that there is a value $c_1 \in (0, 1)$ such that $f'(c_1) = 0$. It also tells us that there is a value $c_2 \in (1, 2)$ such that $f'(c_2) = 0$. But f' is also a differentiable function; and so the MVT applies to it, telling us that for some value $d \in (c_1, c_2)$ we have $f''(d) = 0$.

3

Suppose we have some quadratic approximation $p_2(x)$ for $f(0)$. If we let

$$p_2(x) = c_0 + c_1 x + c_2 x^2$$

then we get

$$\begin{aligned} p_2(0) &= c_0 \\ p_2'(0) &= c_1 \\ p_2''(0) &= 2c_2. \end{aligned}$$

[1] If $n = 0$, then $1/(n\pi)$ does not make sense.

If we want these to agree with f, then we must let $c_0 = f(0)$, $c_1 = f'(0)$, and $c_2 = f''(0)/2$. So our quadratic approximation is

$$p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2.$$

If we do the same for a cubic we get

$$p_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3.$$

and in general

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k$$

As $n \to \infty$, we'd like to say that the approximations get better and better, so the p_n approach f in some way. This turns out to work (for many functions), which is the idea behind Taylor series (which will be explored in much more depth in 31B).