## Related Rates

For these problems, please don't worry about simplifying your answer. The important part is knowing how to set up the problem correctly.

1. Note: the volume of a sphere with radius $r$ is $(4/3)\pi r^3$.

   (a) Argon gas is pumped into a spherical balloon (which starts empty) at a rate of 2 cubic centimeters per second. After 18 seconds, at what rate is the radius of the balloon changing?

   (b) At what rate (as a function of time) do we need to pump argon into the balloon if we want the radius to increase at a constant rate of 5 centimeters per second?

2. (a) Two (perfectly straight) roads diverge in a yellow wood at right angles to one another. Robert picks one of the paths and starts walking at a constant speed of 6 km/h. Three hours later, Frost comes to the same intersection and picks the other path, walking down it at a constant speed of 7 km/h. One hour after Frost leaves the intersection, at what rate is the distance between the two increasing?

   (b) What would the answer be if the angle between the two paths is $\theta$ instead of a right angle, for some $0 \leq \theta \leq \pi/2$? (You'll need a law from trigonometry for this problem. If you don't remember it, feel free to ask.) Is your answer always positive? Why or why not?

## Solutions

### 1(a)

Since the balloon stays spherical, we have the relation

$$V(t) = \frac{4}{3}\pi(r(t))^3$$

where $V(t)$ and $r(t)$ are the volume and radius respectively, which are both functions of $t$. We'll write them as $V$ and $r$ for simplicity. Taking the derivative yields

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

Note that we had to use the chain rule to differentiate the right-hand side. By assumption, $\frac{dV}{dt} = 2$. So we solve for $\frac{dr}{dt}$:

$$\frac{dr}{dt} = \frac{2}{4\pi r^2} = \frac{1}{2\pi r^2}$$

Until this point, everything has been a function of $t$, including $\frac{dV}{dt}$ and $\frac{dr}{dt}$. But we want $\frac{dr}{dt}$ at time $t = 18$, which means we need to find $r$ at time $t = 18$. To do this we go back to the original relation

$$V(18) = \frac{4}{3}\pi(r(18))^3$$

and since the balloon started empty and had argon pumped in at a rate of 2 cc/s, we have $V(18) = 2 \cdot 18 = 36$ cc. So

$$36 = \frac{4}{3}\pi(r(18))^3$$

which means $r(18) = \sqrt[3]{36 \cdot 3/(4\pi)} = 3/\sqrt[3]{\pi}$. If we plug this into the previous equation we get

$$\frac{dr}{dt} = \frac{1}{2\pi(3/\sqrt[3]{\pi})^2} = \frac{1}{18\sqrt[3]{\pi}} \approx 0.1138 \text{ cm/s}$$

### 1(a)

We go back to the equation

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

and fix $\frac{dr}{dt} = 5$. That tells us

$$\frac{dV}{dt} = 20\pi r^2$$

but we want our solution in terms of time, not radius. Fortunately, we know how the radius changes with time: at time $t$, the radius is $5t$. Plug this in and we get

$$\frac{dV}{dt} = \frac{20}{3}\pi(5t)^2 = 2500\pi t^2$$

**2(a)**

Let Robert's distance from the fork in the road be $r$, and let Frosts's distance be $f$, with $x$ the distance between the two walkers. Since the paths define a right triangle, the relation is

$$x^2 = r^2 + f^2$$

Differentiating with respect to time (and remembering to use the chain rule for all three terms) gives

$$2x \cdot \frac{dx}{dt} = 2r \cdot \frac{dr}{dt} + 2f \cdot \frac{df}{dt}$$

The problem gives $\frac{dr}{dt} = 6$ and $\frac{dr}{dt} = 7$. And at time 1, we know that $r = 24$ and $f = 7$. That means $x = \sqrt{7^2 + 24^2} = 25$. So

$$\frac{dx}{dt} = \frac{2 \cdot 24 \cdot 6 + 2 \cdot 7 \cdot 7}{2 \cdot 25} = \frac{288 + 98}{50} = 7.72 \text{ km/h}$$

**2(b)**

Since we don't have a right triangle anymore, we need to use the Law of Cosines:

$$x^2 = r^2 + f^2 - 2rf\cos\theta$$

Now, when we differentiate, we need to notice that $\theta$, and thus $\cos\theta$ is a constant[1]. And we need to use the product rule on the last term since $r$ and $f$ are both functions of $t$. So we get

$$2x \cdot \frac{dx}{dt} = 2r \cdot \frac{dr}{dt} + 2f \cdot \frac{df}{dt} - 2\cos\theta \left( r \cdot \frac{df}{dt} + f \cdot \frac{dr}{dt} \right)$$

Solving for $\frac{dx}{dt}$ gives

$$\begin{aligned} \frac{dx}{dt} &= \frac{r \cdot \frac{dr}{dt} + f \cdot \frac{df}{dt} - \cos\theta \left( r \cdot \frac{df}{dt} + f \cdot \frac{dr}{dt} \right)}{x} \\ &= \frac{r \cdot \frac{dr}{dt} + f \cdot \frac{df}{dt} - \cos\theta \left( r \cdot \frac{df}{dt} + f \cdot \frac{dr}{dt} \right)}{\sqrt{r^2 + f^2 - 2rf\cos\theta}} \\ &= \frac{193 - 210\cos\theta}{\sqrt{625 - 336\cos\theta}} \end{aligned}$$

Note that when $\theta$ is very small, $\frac{dx}{dt}$ is negative. This is because if the paths are very close, Frost is 'catching up' to Robert.

[1]Or equivalently, we can treat it as a variable where $\frac{d\theta}{dt} = 0$.