1. Take the derivative of the following functions:
   (a) $\csc^2 x - \cot^2 x$
   (b) $(x+4)(\cos 3x)$
   (c) $\sin(\cos(\tan(2x)))$

2. Find
$$\lim_{x\to\infty} \frac{\sin x}{x}$$
(Note that this is *not* the limit as $x \to 0$, which we know is 1. Hint: The theorem you want also applies to limits to infinity).

3. Find
$$\lim_{x\to\pm\infty} \frac{x+7}{\sqrt{4x^2-2x+2}}$$
(That is, find both the limit as $x \to \infty$ and the limit as $x \to -\infty$).

4. Find the 100th derivative of xe^x. (If you have a lot of extra time: find the 100th derivative of x^2e^x.) If you're familiar with induction, you can try proving that your answer is correct.

Be ready for the following topics:

- More limits, including those solved by algebraic manipulation
- Definition of the derivative
- Computing tangent lines
- Interpreting the derivative graphically
- Intermediate value theorem

Solutions

1a

We could use the direct approach of using the chain rule on each of the two terms:

$$\begin{aligned}(\csc^2 x - \cot^2 x) &= (\csc^2 x)' - (\cot^2 x)' \\ &= (2\csc x)(-\csc x \cot x) - (2\cot x)(-\csc^2 x) \\ &= -2\csc^2 x \cot x + 2\csc^2 x \cot x \\ &= 0\end{aligned}$$

This is surprising until you remember the trig identity[1] $\csc^2 x - \cot^2 x = 1$; since our function is constant, the derivative must be zero.

1b

We can proceed with the product rule:

$$\begin{aligned}((x+4)(\cos 3x))' &= (x+4)'(\cos 3x) + (x+4)(\cos 3x)' \\ &= \cos 3x + (x+4)(-3\sin 3x)\end{aligned}$$

Note that we could also have expanded and then taken the derivative:

$$\begin{aligned}((x+4)(\cos 3x))' &= (x\cos 3x + 4\cos 3x)' \\ &= \cos 3x - 3x\sin 3x - 12\sin 3x\end{aligned}$$

which is the same answer.

1c

We'll have to use the chain rule several times:

$$\begin{aligned}(\sin(\cos(\tan(2x))))' &= \cos(\cos(\tan(2x)))(\cos(\tan(2x)))' \\ &= \cos(\cos(\tan(2x))) \cdot (-\sin(\tan(2x)) \cdot (\tan(2x))' \\ &= \cos(\cos(\tan(2x))) \cdot (-\sin(\tan(2x)) \cdot 2\sec^2 2x\end{aligned}$$

[1] If we write cosecant and cotangent as fractions, this becomes clearer:

$$\frac{1}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x} = \frac{\sin^2 x}{\sin^2 x}$$

which is just $\sin^2 x + \cos^2 x = 1$, divided by $\sin^2 x$ and rearranged.

2

This is a prototypical Squeeze Theorem example. Since $-1 \leq \sin x \leq 1$, we can multiply by $1/x$ to get

$$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

Now as $x \to \infty$, both the lower bound (the left-hand side) and the upper bound (the right-hand side) go to zero. So by the Squeeze Theorem, the limit must be zero.

3

If we divide out the top and bottom by x, we get

$$\lim_{x \to \pm\infty} \frac{x+7}{\sqrt{4x^2 - 2x + 2}} = \lim_{x \to \pm\infty} \frac{1 + 7/x}{\frac{1}{x} \cdot \sqrt{4x^2 - 2x + 2}}$$

We'd like to pull the $1/x$ into the square root in the denominator, but what happens depends on the sign of x, so we'll split this up[2].

$$\begin{aligned}\lim_{x \to +\infty} \frac{1 + 7/x}{\frac{1}{x} \cdot \sqrt{4x^2 - 2x + 2}} &= \lim_{x \to +\infty} \frac{1 + 7/x}{\sqrt{4 - 2/x + 2/x^2}} \\ &= \frac{1+0}{\sqrt{4 - 0 - 0}} \\ &= \frac{1}{2}\end{aligned}$$

but

$$\begin{aligned}\lim_{x \to -\infty} \frac{1 + 7/x}{\frac{1}{x} \cdot \sqrt{4x^2 - 2x + 2}} &= \lim_{x \to -\infty} \frac{1 + 7/x}{-\sqrt{4 - 2/x + 2/x^2}} \\ &= \frac{1+0}{-\sqrt{4 - 0 - 0}} \\ &= -\frac{1}{2}\end{aligned}$$

4

Let's take a few derivatives and look for a pattern:

[2]This is because

$$\sqrt{1/x^2} = |1/x|$$

which is equal to $1/x$ if x is positive but is equal to $-1/x$ when x is negative.

$$\begin{aligned} f(x) &= xe^x \\ f'(x) &= xe^x + e^x \\ f''(x) &= xe^x + 2e^x \\ f'''(x) &= xe^x + 3e^x \end{aligned}$$

We can see that at every step the xe^x term spits out another e^x, and that the e^x term doesn't do anything. So it looks like the nth derivative is $xe^x + ne^x$, and we can plug in 100 if we want[3].

As for x^2e^x, we can again take some derivatives:

$$\begin{aligned} f(x) &= x^2e^x \\ f'(x) &= x^2e^x + 2xe^x \\ f''(x) &= x^2e^x + 4xe^x + 2e^x \\ f'''(x) &= x^2e^x + 6xe^x + (2+4)e^x \\ f^{(4)}(x) &= x^2e^x + 8xe^x + (2+4+6)e^x \end{aligned}$$

So we can guess that $f^{(n)}(x) = x^2e^x + 2nxe^x + (2+4+\cdots+2(n-1))e^x$. Now

$$2+4+\cdots+2(n-1) = 2(1+2+\cdots+(n-1)) = 2\cdot\frac{(n-1)n}{2} = (n-1)(n)$$

meaning that our derivative is

$$f^{(n)}(x) = x^2e^x + 2nxe^x + n(n-1)e^x$$

and if we plug in $n = 100$ we get

$$x^2e^x + 200xe^x + 9900e^x$$

[3] An inductive proof might look like this: we claim $f^{(n)}(x) = xe^x + ne^x$ for all $n \geq 0$. We've already checked the case $n = 0$ (as well as $n = 1, 2, 3$) above.

If it is true that $f^{(k)}(x) = xe^x + ke^x$, then

$$\begin{aligned} f^{(k+1)} &= \left(f^{(k)}\right)' \\ &= (xe^x + ke^x)' \\ &= xe^x + e^x + ke^x \\ &= xe^k + (k+1)e^x \end{aligned}$$

So induction tells us that we have the general formula.