1. Find tangent lines to the following functions at the prescribed points:

   (a) $g(x) = \frac{x^2-1}{x^2+1}$ at $x = 3$.

2. Compute the derivatives of the following functions using the limit definition:

   (a) $x^3 + 8$.

   (b) $\sqrt{x}$ (Hint: This one may be easier using the other limit form.)

   (c) $\sin x$ (Hint: Recall the trig identity $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, and use the special trig limits we talked about last week.)

3. Find 10 functions whose derivative is x^2.

4. Find the analogue of the product rule when there are *three* functions; that is, what is

$$(f_1 f_2 f_3)'$$

   when f_1, f_2, f_3 are differentiable functions? See if you can find a general formula for when there are n factors.

Solutions

1(a)

This is already a line! So the tangent line is just $e^{\sqrt{\pi}}$.

1(b)

We compute $g'(x)$:

$$g'(x) = \frac{2x(x^2+1) - 2x(x^2-1)}{(x^2+1)^2} = \frac{2x - (-2x)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

So we can compute

$$\begin{aligned} g(3) &= \frac{3^2-1}{3^2+1} = \frac{8}{10} = \frac{4}{5} \\ g'(3) &= \frac{4 \cdot 3}{(3^2+1)^2} = \frac{12}{100} = \frac{3}{25} \end{aligned}$$

which means the tangent line is

$$\frac{4}{5} + \frac{3}{25}(x-3)$$

1(c)

Rather than compute h' completely, we can just know that h' is going to be a polynomial (since it's the derivative of a polynomial) and it will have constant term 2. So $h'(0) = 2$, as all the other terms go away when you plug in zero. Similarly, $h(0) = 1$. So the tangent line is $2x + 1$.

2(a)

We set up the limit:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^3 + 8 - (x^3+8)}{h} \\ &= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 8 - x^3 - 8}{h} \\ &= \lim_{h \to 0} 3x^2 + 3xh + h^2 \\ &= 3x^2 \end{aligned}$$

2(b)

Let's do the limit in the other way:

$$\begin{aligned} f'(x) &= \lim_{y\to x} \frac{f(y)-f(x)}{y-x} \\ &= \lim_{y\to x} \frac{\sqrt{y}-\sqrt{x}}{y-x} \\ &= \lim_{y\to x} \frac{\sqrt{y}-\sqrt{x}}{(\sqrt{y}-\sqrt{x})(\sqrt{y}+\sqrt{x})} \\ &= \lim_{y\to x} \frac{1}{\sqrt{y}+\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

2(c)

Here we use the sum-of-sines formula, plus the fact that $\lim_{h\to 0} \sin h/h = 1$ and $\lim_{h\to 0}(\cos h - 1)/h = 0$:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\ &= \lim_{h\to 0} \left(\frac{\sin(x)(\cos(h)-1)}{h} + \frac{\cos(x)\sin(h)}{h} \right) \\ &= \sin(x)\cdot \lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos(x)\cdot \lim_{h\to 0} \frac{\sin(h)}{h} \\ &= \sin(x)\cdot 0 + \cos(x)\cdot 1 \\ &= \cos(x) \end{aligned}$$

3

It feels like the answer should be a monomial, and if we look at $(cx^n)' = cnx^{n-1} = x^2$, we get $n = 3$ and $c = 1/3$. So $x^3/3$ works; for other solutions, note that moving the graph up and down will not change slopes, so $x^3/3 + C$ works for any C. (We can also see this by using the sum rule).

Are there any others? If we have a function f such that $f' = x^2$, then

$$(f - x^3/3)' = f' - x^2 = x^2 - x^2 = 0$$

which means $f - x^3/e$ is a constant. So these are the only ones.

4

We apply the product rule twice:

$$\begin{aligned}(f_1 f_2 f_3)' &= f_1'(f_2 f_3) + f_1(f_2 f_3)' \\ &= f_1' f_2 f_3 + f_1(f_2' f_3 + f_2 f_3') \\ &= f_1' f_2 f_3 + f_1 f_2' f_3 + f_1 f_2 f_3'\end{aligned}$$

If we had n factors, we would have a similar thing with n terms, each of which has one derivative.