1. Evaluate the following limits:

   (a)
   $$\lim_{x\to -4} \frac{x^2+3x-4}{(x+4)^3}$$

   (b)
   $$\lim_{x\to 0} \frac{x^2}{\sin(x)}$$

   (c)
   $$\lim_{x\to 2} \frac{x-2}{\sqrt{x}-\sqrt{2}}$$

   (d)
   $$\lim_{x\to 0} \frac{\sin(x)}{\sin(\sin(x))}$$

   (e)
   $$\lim_{x\to 1} x^{\left(2+\frac{1}{\ln x}\right)}$$

2. (a) Compute
   $$\lim_{t\to 0} \quad (\sin t)\cdot(2\sin(1/t)-\cos(1/t^3)+8)$$
   (Hint: there is a theorem that will help you).

   (b) Suppose we have a limit of the form
   $$\lim_{x\to a} f(x)\cdot g(x)$$
   where
   $$\lim_{x\to a} f(x)=0 \qquad \text{and} \qquad |g(x)|<M \text{ for all } x.$$
   for some number M. Find the limit[1].

[1] Actually, all we need is for $|g(x)|<M$ to hold close to a; that is, $|g(x)|<M$ whenever $0<|x-a|<\varepsilon$ for some $\varepsilon>0$.

Solutions

1(a)

Factoring the top yield the limit

$$\begin{aligned}\lim_{x\to -4}\frac{x^2+3x-4}{(x+4)^3} &= \lim_{x\to -4}\frac{(x+4)(x-1)}{(x+4)^3}\\ &= \lim_{x\to -4}\frac{x-1}{(x+4)^2}\end{aligned}$$

Now we have a fraction where the denominator goes to zero while the numerator is bounded; thus the limit cannot exist. We can say a little more in this case, though. On either side of -4, the numerator is negative while the denominator is positive, so we are going to $-\infty$ on both sides. Thus we can also say that the limit is $-\infty$.

1(b)

We know about the limit of $\sin x/x$, so let's turn this into that:

$$\begin{aligned}\lim_{x\to 0}\frac{x^2}{\sin(x)} &= \lim_{x\to 0}\left(\frac{x}{\sin(x)}\cdot x\right)\\ &= \lim_{x\to 0}\frac{x}{\sin(x)}\cdot\lim_{x\to 0} x\\ &= 1\cdot 0 = 0\end{aligned}$$

Note that the step going from the limit of a product to the product of limits is valid because both of the factors' limits exist.

1(c)

This isn't quite a rational function, but we can deal with it in the same way, by factoring the numerator:

$$\begin{aligned}\lim_{x\to 2}\frac{x-2}{\sqrt{x}-\sqrt{2}} &= \lim_{x\to 2}\frac{(\sqrt{x}-\sqrt{2})(\sqrt{x}+\sqrt{2})}{\sqrt{x}-\sqrt{2}}\\ &= \lim_{x\to 2}\left(\sqrt{x}+\sqrt{2}\right)\\ &= 2\sqrt{2}\end{aligned}$$

(We could write this in a more familiar form by letting $y=\sqrt{x}$; then the limit becomes

$$\lim_{y\to\sqrt{2}}\frac{y^2-2}{y-\sqrt{2}}$$

which is a rational function).

1(d)

Here the way forward is a substitution: if we let $y = \sin(x)$ we get

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(x)}{\sin(\sin(x))} &= \lim_{y\to \sin 0} \frac{y}{\sin y} \\ &= \lim_{y\to 0} \frac{y}{\sin y} = 1\end{aligned}$$

1(e)

The best way to deal with expressions involving exponents (especially when there are variables in both base and exponent) is to take logarithms:

$$\begin{aligned}\ln\left(\lim_{x\to 1} x^{\left(2+\frac{1}{\ln x}\right)}\right) &= \lim_{x\to 1} \ln\left(x^{\left(2+\frac{1}{\ln x}\right)}\right) \\ &= \lim_{x\to 1}\left(2+\frac{1}{\ln x}\right)\cdot \ln x \\ &= \lim_{x\to 1} 2\ln x + 1 \\ &= 1\end{aligned}$$

We are allowed to bring the logarithm inside the limit because ln is a continuous function. Since the logarithm of our limit is 1, the limit must be e.

2(a)

The way forward is with the squeeze theorem. Now, if we look at the piece of the limit in brackets, we can say

$$-2019 \le (2\sin(1/t) - \cos(1/t^3) + 8) \le 2019$$

(Of course, these are not the only bounds you could have, and they're definitely not the 'best'. But that doesn't really affect the argument). So

$$-2019|\sin t| \le (\sin t)\cdot(2\sin(1/t) - \cos(1/t^3) + 8) \le 2019|\sin t|$$

Both the left and right side functions go to zero as $t \to 0$, and by the Squeeze Theorem this means the limit is zero.

2(b)

Since $|g| \le M$, we have

$$-|f|\cdot M \le f\cdot g \le |f|\cdot M$$

(The absolute values are there to account for cases where $f < 0$). Now if $f \to 0$, so does $|f|$. Thus both sides go to zero, and by the Squeeze Theorem the limit of $f \cdot g$ is also zero.