Induction

Recall the principle of mathematical induction: Suppose we have a statement $P(n)$. For instance, we could want to prove that the sum of the first n odd integers is n^2. Then $P(n)$ is the statement $1 + 3 + \cdots + (2n - 1) = n^2$.

To prove it for all integers $n \geq a$, we just need to establish:

- The *base case*: Show that $P(a)$ is true.
- The *inductive step*: Show that $P(k)$ being true implies that $P(k + 1)$ is true, for any $k \geq a$

For each of the following, identify the variable we are inducting on. Then write the base case as well as the inductive step. Use this to prove these statements by induction.

1. In the famous song "The Twelve Days of Christmas," the singer's lover gives them 1 gift on the first day, then 1+2 gifts on the second day, 1+2+3 gifts on the third day, and so on. The total number of gifts received up to and including day n is

$$\frac{n(n+1)(n+2)}{6}, \text{ or } \binom{n+2}{3}$$

2. For any integer $k \geq 7$,

$$3^k < k!$$

3. For any positive integer n,

$$(1 + 2 + \cdots + n)^2 = 1^3 + 2^3 + \cdots + n^3$$

Solutions

1

We induct on n. The base case occurs when $n = 1$; on the first day, the singer receives one gift, which is

$$\frac{1 \cdot (1+1) \cdot (1+2)}{6} = \frac{6}{6} = 1.$$

So suppose we've received the correct number of gifts up to day n. Then we receive $1 + \cdots + (n+1)$ gifts on the $n+1$st day. That's equal to $\frac{(n+1)(n+2)}{2}$, so we have gotten

$$\begin{aligned} &\frac{n(n+1)(n+2)}{6} + \frac{(n+1)(n+2)}{2} \\ &= \frac{n(n+1)(n+2) + 3(n+1)(n+2)}{6} \\ &= \frac{(n+3)(n+1)(n+2)}{6} \end{aligned}$$

by the $n+1$st day, as required. Note that we can get a slightly nicer argument using binomial coefficients: by the nth day, we've received $\binom{n+2}{3}$, and on the $n+1$st day we get $\frac{(n+2)(n+1)}{2} = \binom{n+2}{2}$. And we know

$$\binom{n+2}{3} + \binom{n+2}{2} = \binom{n+3}{3}$$

by binomial identities. The numbers we computed here are called *tetrahedral numbers* because the nth one is the number of spheres needed to make a regular tetrahedron with side length n.

2

We induct on k. The base case occurs when $n = 7$; we need to check

$$3^7 = 2187 < 5040 = 7!$$

which is true.

Now for the inductive step, we get to assume

$$3^k < k!$$

and, since $k + 1 > 3$ (and everything is positive), we can multiply the left hand side by 3 and the right hand side by $k + 1$ to get

$$3^k \cdot 3 < k! \cdot k + 1$$

which says exactly that $3^{k+1} < (k+1)!$, as required.

3

We induct on n. The base case occurs when $n = 0$; then we want

$$(1)^2 = 1^3$$

which is clearly true. Now for the inductive step: we are able to assume

$$(1 + 2 + \cdots + n)^2 = 1^3 + \cdots + n^3 \tag{1}$$

and want to show

$$(1 + 2 + \cdots + n + (n+1))^2 = 1^3 + \cdots + n^3 + (n+1)^3 \tag{2}$$

Suppose we subtract equation (1) from (2): then we get

$$\begin{aligned}(1 + 2 + \cdots + n + (n+1))^2 - (1 + 2 + \cdots + n)^2 &\stackrel{?}{=} (n+1)^3\\ 2 \cdot (n+1)(1 + \cdots + n) + (n+1)^2 &\stackrel{?}{=} (n+1)^3\\ 2 \cdot (n+1) \cdot \frac{n(n+1)}{2} + (n+1)^2 &\stackrel{?}{=} (n+1)^3\\ (n+1)^2 \cdot (n+1) &\stackrel{?}{=} (n+1)^3\end{aligned}$$

So if we *add* this difference to both sides of (1) we get (2) as required.