Induction

Recall the principle of mathematical induction: Suppose we have a statement $P(n)$. For instance, we could want to prove that the sum of the first n odd integers is n^2. Then $P(n)$ is the statement $1 + 3 + \cdots + (2n - 1) = n^2$.

To prove it for all integers $n \geq a$, we just need to establish:

- The *base case*: Show that $P(a)$ is true.
- The *inductive step*: Show that $P(k)$ being true implies that $P(k + 1)$ is true, for any $k \geq a$

For each of the following, identify the variable we are inducting on. Then write the base case as well as the inductive step. Use this to prove these statements by induction.

1. For any positive integer n,

$$2^0 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

2. Bernoulli's inequality: for any integer $k \geq 0$ and any real number $r > -1$,

$$(1 + r)^k \geq 1 + kr$$

3. For any positive integer n,

$$(1 + 2 + \cdots + n)^2 = 1^3 + 2^3 + \cdots + n^3$$

Solutions

1

We induct on n. The base case occurs when $n = 1$:

$$2^0 + 2^1 \stackrel{?}{=} 2^{1+1} - 1$$
$$1 + 2 = 4 - 1$$

which is true. Now, for the inductive step, we get to assume

$$2^0 + 2^1 + \cdots + 2^k = 2^{k+1} - 1 \tag{1}$$

and we'd like to prove

$$2^0 + 2^1 + \cdots + 2^k + 2^{k+1} = 2^{k+2} - 1. \tag{2}$$

Now we can turn the left-hand side of (1) into the left-hand side of (2) just by adding 2^{k+1} to both sides of (1):

$$\begin{aligned} 2^0 + 2^1 + \cdots + 2^k + 2^{k+1} &= 2^{k+1} - 1 + 2^{k+1} \\ &= 2^{k+2} - 1 \end{aligned}$$

This is exactly (2), which is what we wanted to show. So by induction, we are done.

2

We induct on k. The base case occurs when $k = 0$; we want that for any $r > -1$,

$$(1 + r)^0 \geq 1 + 0r$$

Since both sides of this equality are 1, this is true. Now our inductive hypothesis is that for any r,

$$(1 + r)^\ell \geq 1 + \ell r \tag{3}$$

and we would like to prove

$$(1 + r)^{\ell+1} \geq 1 + (\ell + 1)r. \tag{4}$$

This time it makes sense to *multiply* both sides of (3) by $(1 + r)$, since that does what we want to the left-hand side. (Note that this is OK since $1 + r > 0$). Then we get

$$\begin{aligned} (1 + r)^\ell \cdot (1 + r) &\geq (1 + \ell r) \cdot (1 + r) \\ (1 + r)^{\ell+1} &\geq 1 + \ell r + r + \ell r^2 \\ &= 1 + (\ell + 1)r + \ell r^2 \\ &\geq 1 + (\ell + 1)r \end{aligned}$$

since, by assumption, $\ell r^2 \geq 0$. This is exactly (4); so, by induction, we are done.

3

We induct on n. The base case occurs when $n = 0$; then we want

$$(1)^2 = 1^3$$

which is clearly true. Now for the inductive step: we are able to assume

$$(1 + 2 + \cdots + n)^2 = 1^3 + \cdots + n^3 \tag{5}$$

and want to show

$$(1 + 2 + \cdots + n + (n+1))^2 = 1^3 + \cdots + n^3 + (n+1)^3 \tag{6}$$

Suppose we subtract equation (5) from (6): then we get

$$\begin{aligned}
(1 + 2 + \cdots + n + (n+1))^2 - (1 + 2 + \cdots + n)^2 &\stackrel{?}{=} (n+1)^3 \\
2 \cdot (n+1)(1 + \cdots + n) + (n+1)^2 &\stackrel{?}{=} (n+1)^3 \\
2 \cdot (n+1) \cdot \frac{n(n+1)}{2} + (n+1)^2 &\stackrel{?}{=} (n+1)^3 \\
(n+1)^2 \cdot (n+1) &\stackrel{?}{=} (n+1)^3
\end{aligned}$$

as required.