Systems of linear equations and rank

1. (Warmup) Solve the following system of equations:

$$\begin{aligned} x + y + z &= 3 \\ x + 2y + 3z &= 6 \\ 2x - 2z &= 0 \end{aligned}$$

   (Answer: $[1, 1, 1] + t[1, -2, 1]$ for any real number t.).

2. What is the rank of the matrix in the previous problem?

3. What is the rank of the matrix

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \\ 20 & 21 & 22 & 23 & 24 \end{bmatrix}$$

   (Hint: If you encounter fractions, you are making the problem harder than it is).

4. (Do only if you have covered dot products[1] already). Compute $[3, -4] \bullet [1, 2]$. Now describe the set of vectors $\mathbf{v}$ such that

$$\mathbf{v} \bullet [1, 2] = 0.$$

5. (Also, do only if you have covered dot products). Find a vector $\mathbf{v}$ such that

$$\begin{aligned} \mathbf{v} \bullet [-1, -2] &= -3 \\ \mathbf{v} \bullet [6, 7] &= 8 \end{aligned}$$

6. Suppose we have a system of a linear equations in b unknowns with a unique solution. What is the rank of the associated matrix? What can we say about the relation between a and b?

[1] Recall that the dot product of two real vectors $\mathbf{a} = [a_1, a_2, \ldots, a_n]$ and $\mathbf{b} = [b_1, b_2, \ldots, b_n]$ is

$$\mathbf{a} \bullet \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Solutions

1

Let's write this as a matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 6 \\ 2 & 0 & -2 & 0 \end{array}\right]$$

Then we row-reduce (I've boxed the pivots at each step):

$$\begin{aligned} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 3 \\ 1 & 2 & 3 & 6 \\ 2 & 0 & -2 & 0 \end{array}\right] &\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & \boxed{1} & 2 & 3 \\ 0 & -2 & -4 & -6 \end{array}\right] \\ &\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ &\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{aligned}$$

If we solve these equations for their leading variable we get

$$\begin{aligned} x &= z \\ y &= 3 - 2z \end{aligned}$$

We can parameterize this line by letting $s = z$; then we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s \\ 3 - 2s \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} s$$

Note that this line is the same as the one given (we've just changed the parameter).

2

There were two pivots, so the rank is 2.

3

Instead of doing a full row-reduction, let's just subtract the first row from all the other rows:

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 5 & 5 & 5 & 5 & 5 \\ 10 & 10 & 10 & 10 & 10 \\ 15 & 15 & 15 & 15 & 15 \\ 20 & 20 & 20 & 20 & 20 \end{bmatrix}$$

Now we can see that the last 3 rows will be killed by subtracting multiples of the second row, and the first two rows are not redundant. So we get that the matrix has rank 2.

4

By definitions,

$$[3, -4] \bullet [1, 2] = 3 \cdot 1 + (-4) \cdot 2 = 3 - 8 = -5$$

If a vector $[x, y]$ has a zero dot product with $[1, 2]$, then

$$x + 2y = 0$$

or $y = -\frac{1}{2}x$. This is the line in the plane which is *perpendicular*[2] to $[1, 2]$.

5

If we have a vector $[v_1, v_2]$ with the given dot products then

$$\begin{aligned} -v_1 - 2v_2 &= -3 \\ 6v_1 + 7v_2 &= 8 \end{aligned}$$

Solving this system gives $[v_1, v_2] = [-1, 2]$.

6

If there is a unique solution, there is a pivot in every column, which means the rank is exactly b. Since this means we must have at least as many rows as columns (as there is at most one pivot per column) that means $a \geq b$.

[2]Maybe an easier way to see this is by using the formula for the angle θ between vectors $\mathbf{v}$ and $\mathbf{w}$:

$$\cos\theta = \frac{\mathbf{v} \bullet \mathbf{w}}{||\mathbf{v}|| \cdot ||\mathbf{w}||}$$

where $||\mathbf{v}||$ is the length or magnitude of $\mathbf{v}$ and is given by $\sqrt{\mathbf{v} \bullet \mathbf{v}}$.