Systems of linear equations

1. (Warmup) Solve the following system of equations:

$$\begin{aligned} x + y + z &= 0 \\ -x + 3z &= 2 \\ 2x + y + z &= 1 \end{aligned}$$

   (Answer: $x = 1$, $y = -2$, $z = 1$).

2. For what values of a, b, and c does the following system of equations have solutions? And how many solutions does it have in the case that it does?

$$\begin{aligned} 2x - y - 3z &= a \\ -x + y - z &= b \\ 5x - 3y - 5z &= c \end{aligned}$$

3. Consider the following system of equations:

$$\begin{aligned} x - 2y + z &= 1 \\ x + y - 2z &= 1 \\ -2x + y + z &= 1 \end{aligned}$$

   How many solutions does this system have? For any pair of the equations, how many solutions does that pair have? What does this system of equations look like graphically?

Solutions

1

Let's write this as a matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1 & 0 & 3 & 2 \\ 2 & 1 & 1 & 1 \end{array}\right]$$

To solve this, let's use the 1 in the top left corner as a pivot to clear the first column. We add the first row to the second, and subtract twice the first row from the third to obtain

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 4 & 2 \\ 0 & -1 & -1 & 1 \end{array}\right]$$

Now we use the 1 in the middle to clear the second column; add the second row to the third row to obtain

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

and then divide the last row by 3 to get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

At this point we are in *row-echelon form*. Now we change to *back-substitution*; subtract the third row from the first and subtract four times the third row from the second to get

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Lastly, we subtract the second row from the first to get

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

as a matrix in *reduced row-echelon form*. We can now read off the solution: $x = 1$, $y = -2$, $z = 3$.

2

We can start with any equation to solve the system; so let's add twice the second equation to the first and five times the second equation to the third:

$$\begin{aligned} y - 5z &= a + 2b \\ -x + y - z &= b \\ 2y - 10z &= c + 5b \end{aligned}$$

Now notice that the first equation and the third are similar; if we subtract twice the first equation from the third we get

$$\begin{aligned} y - 5z &= a + 2b \\ -x + y - z &= b \\ 0 &= c + 5b - 2(a + 2b) \end{aligned}$$

The last equation tells us that $-2a + b + c = 0$; if we have this, then the first two equations are always satisfiable. So we get that the solution set is a line when $-2a + b + c = 0$, and empty otherwise.

3

Note that if we add the three equations together, we get $0 = 3$, so there's no solutions to the system. If we look at any pair of equations, then they intersect in a line. So the picture here is three planes that intersect in three distinct parallel lines.