

Toroidal compactifications of the moduli spaces of Drinfeld modules, I

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1 Introduction

1.1 Summary

1.1.1. Let F be a function field in one variable over a finite field. Fix a place ∞ of F , and let A be the subring of F consisting of all elements which are integral outside ∞ .

In this paper (Part I) and its sequel (Part II), we construct toroidal compactifications of the moduli spaces of Drinfeld A -modules. In this Part I, we construct them in the case $A = \mathbb{F}_q[T]$. In Part II, we will construct them for general A by a method of reduction to the case $A = \mathbb{F}_q[T]$.

1.1.2. In this paper, we refer to Drinfeld A -modules more simply as Drinfeld modules.

Assume $A = \mathbb{F}_q[T]$, and let N be an element of A which does not belong to \mathbb{F}_q .

In the case that N has at least two distinct prime divisors (resp. N has only one prime divisor), let \mathcal{M}_N^d over A (resp. over $A[\frac{1}{N}]$) be the moduli space of Drinfeld modules of rank d with Drinfeld level N structure. Recall that by [5, Section 5], the moduli space \mathcal{M}_N^d is regular, and $\mathcal{M}_N^d \otimes_A A[\frac{1}{N}]$ is smooth over $A[\frac{1}{N}]$.

In this paper, we construct toroidal compactifications of \mathcal{M}_N^d as the moduli spaces of *log Drinfeld modules of rank d with level N structure*.

1.1.3. Similarly to toroidal compactifications of the moduli spaces of abelian varieties as in [3, 6], our toroidal compactifications are indexed by cone decompositions and they form a projective system under subdivisions. After inverting N , they are log smooth over $A[\frac{1}{N}]$, and among them, smooth toroidal compactifications are cofinal. In the case N has at least two prime divisors, toroidal compactifications (without inverting N) are log regular (or equivalently, log smooth over \mathbb{F}_q by 2.7.2), and among them, regular toroidal compactifications are cofinal.

1.1.4. The compactification of \mathcal{M}_N^d of Satake-Baily-Borel type (called a Satake compactification in [25]) was constructed by Kapranov [18] in the case that $A = \mathbb{F}_q[T]$ and by Pink [25] in general.

In a short summary [24], Pink explained how to construct toroidal compactifications of \mathcal{M}_N^d quoting a work of K. Fujiwara. The details are not yet published. In this paper, we use the ideas of Pink and Fujiwara.

The toroidal compactification of the moduli space of Drinfeld modules with $A = \mathbb{F}_q[T]$ and $N = T$ is contained in the work of Puttick [28].

In the case $d = 2$, our toroidal compactification coincides with the proper model of the Drinfeld modular curve constructed in Lehmkuhl [23].

1.2 Log Drinfeld modules with level structures

Let F , ∞ , and A be as in 1.1.1. Let p be the characteristic of F .

1.2.1. Let S be a scheme over A .

Recall that a Drinfeld module over S of rank $d \geq 1$ is a pair (\mathcal{L}, ϕ) where \mathcal{L} is a line bundle over S and ϕ is an action of A on the additive group scheme \mathcal{L} , satisfying a certain condition. A generalized Drinfeld module over S is such a pair (\mathcal{L}, ϕ) satisfying a slightly weaker condition, found in 2.1.

The Satake compactification of \mathcal{M}_N^d obtained in [18] and [25] is regarded as the moduli space of generalized Drinfeld modules (see Section 5.1).

For the toroidal compactification, we introduce the notion of a log Drinfeld module of rank d with level N structure below.

1.2.2. Let N be an element of A which does not belong to the total constant field of F . Let $d \geq 1$ be an integer.

Recall that for a scheme S over A and for a Drinfeld A -module (\mathcal{L}, ϕ) over S of rank d , a Drinfeld level N structure on (\mathcal{L}, ϕ) is a homomorphism

$$\iota: (\frac{1}{N}A/A)^d \rightarrow \mathcal{L},$$

which is compatible with the actions of A , where A acts on \mathcal{L} via ϕ , and which satisfies a certain condition (2.6.1). If N is invertible on S , the condition is that ι induces an isomorphism

$$(\frac{1}{N}A/A)^d \xrightarrow{\sim} \phi[N]$$

of group schemes, where $\phi[N] = \ker(\phi(N): \mathcal{L} \rightarrow \mathcal{L})$.

1.2.3. In this paper, we consider schemes with log structures. For generalities of log structures, see [19] and [15]. See 2.5.1 of this paper for a short introduction. A scheme endowed with a log structure is called a log scheme. In this paper, we use saturated log structures (2.5.1). Saturated log structures have good relations with toric geometry and cone decompositions which we use in our theory of toroidal compactifications.

1.2.4. A standard example of a saturated log structure which we consider is as follows.

Let S be a normal scheme and let U be a dense open subset of S with the inclusion morphism $j: U \rightarrow S$. Then the sheaf $M_S := \mathcal{O}_S \cap j_*(\mathcal{O}_U^\times) \subset j_*(\mathcal{O}_U) = \{f \in \mathcal{O}_S \mid f \text{ is invertible on } U\}$ with the map $M_S \xrightarrow{\hookrightarrow} \mathcal{O}_S$ is a saturated log structure. We call M_S the log structure on S associated to U (or the log structure on S associated to the closed set $S \setminus U$).

1.2.5. Our toroidal compactifications are related to toric varieties. Consider the affine toric variety $\text{Spec}(\mathbb{F}_p[P])$ over \mathbb{F}_p for a finitely generated saturated monoid (fs monoid for short, see 2.5.1) P , where $\mathbb{F}_p[P]$ denotes the semi-group ring of P over \mathbb{F}_p . It has the standard log structure.

For a scheme S over A endowed with a log structure M_S , the following four conditions (i)–(iv) are equivalent.

- (i) S is log smooth over \mathbb{F}_p .
- (ii) S is log regular (2.7.1) and locally of finite type over A .
- (iii) Étale locally on S , there are an fs monoid P and an étale morphism $S \rightarrow \text{Spec}(\mathbb{F}_p[P])$ such that M_S is the inverse image of the standard log structure of $\text{Spec}(\mathbb{F}_p[P])$.
- (iv) Étale locally on S , there are an fs monoid P and a smooth morphism $S \rightarrow \text{Spec}(\mathbb{F}_p[P])$ such that M_S is the inverse image of the standard log structure of $\text{Spec}(\mathbb{F}_p[P])$.

Under these equivalent conditions, S is normal and $M_S = \mathcal{O}_S \cap j_*(\mathcal{O}_U^\times)$ as in 1.2.4, where U is the dense open set of S consisting of all points at which the log structure of S is trivial.

1.2.6. Let S be a scheme with a saturated log structure M_S . For a line bundle \mathcal{L} on S , we define a sheaf $\overline{\mathcal{L}}$ on the étale site of S which contains \mathcal{L} as a subsheaf. Put succinctly, $\overline{\mathcal{L}}$ is an enlargement of \mathcal{L} allowing poles which belong to the log structure of S . It plays a role here because torsion points of generalized Drinfeld modules have poles. Define

$$\overline{\mathcal{L}} := \mathcal{L} \cup_{\mathcal{L}^\times} M_S^{-1} \mathcal{L}^\times.$$

Here $\mathcal{L}^\times \subset \mathcal{L}$ denotes the sheaf of bases of \mathcal{L} ,

$$M_S^{-1} = \{f^{-1} \mid f \in M_S\} \subset M_S^{\text{gp}} = \{fg^{-1} \mid f, g \in M_S\},$$

$M_S^{-1}\mathcal{L}^\times$ denotes the quotient $M_S^{-1} \times_{\mathcal{O}_S^\times} \mathcal{L}^\times$ of $M_S^{-1} \times \mathcal{L}^\times$ by the relation $(hu, e) \sim (h, ue)$ ($h \in M_S^{-1}$, $u \in \mathcal{O}_S^\times$, $e \in \mathcal{L}^\times$), and $\cup_{\mathcal{L}^\times}$ means the union obtained by identifying the subsheaf \mathcal{L}^\times of \mathcal{L} and the subsheaf \mathcal{L}^\times of $M_S^{-1}\mathcal{L}^\times$.

For (S, U) as in 1.2.4 and for the associated log structure M_S , $\overline{\mathcal{L}} = \mathcal{L} \cup M_S^{-1}\mathcal{L}^\times \subset j_*(\mathcal{L}|_U)$.

Note that the additive group structure of \mathcal{L} does not extend to $\overline{\mathcal{L}}$.

If \mathcal{L} is the trivial line bundle \mathcal{O}_S , then $\overline{\mathcal{L}}$ is identified with the union $\mathcal{O}_S \cup_{\mathcal{O}_S^\times} M_S^{-1}$, and it is identified with the sheaf of morphisms to the log scheme $\mathbf{P}_{\mathbb{Z}}^1$, the projective line over \mathbb{Z} , with the log structure associated to the divisor at infinity.

Let

$$\text{pole}: \overline{\mathcal{L}} \rightarrow M_S/\mathcal{O}_S^\times$$

be the map which sends \mathcal{L} to 1 and sends $f^{-1}e$ to $f \bmod \mathcal{O}_S^\times$ for $f \in M_S$ and $e \in \mathcal{L}^\times$.

1.2.7. Let (S, U) be as in 1.2.4 and assume that S is a scheme over A . Then by a generalized Drinfeld module over (S, U) of rank d with level N structure, we mean a pair $((\mathcal{L}, \phi), \iota)$ of a generalized Drinfeld module (\mathcal{L}, ϕ) over S whose restriction $(\mathcal{L}, \phi)|_U$ to U is a Drinfeld module of rank d and ι is a level N structure on $(\mathcal{L}, \phi)|_U$.

We can show (2.6.4) that for such (S, U) and $((\mathcal{L}, \phi), \iota)$, $\iota: (\frac{1}{N}A/A)^d \rightarrow \mathcal{L}|_U$ extends uniquely to a map $\iota: (\frac{1}{N}A/A)^d \rightarrow \overline{\mathcal{L}} \subset j_*(\mathcal{L}|_U)$, where $\overline{\mathcal{L}}$ is defined by the log structure of S in 1.2.4.

1.2.8. Let S be a scheme over A with saturated log structure. By a *log Drinfeld module* over S of rank d with level N structure, we mean a pair $((\mathcal{L}, \phi), \iota)$ of a generalized Drinfeld module (\mathcal{L}, ϕ) over S and a map

$$\iota: (\frac{1}{N}A/A)^d \rightarrow \overline{\mathcal{L}}$$

satisfying a certain condition (i)

(i) Étale locally on S , there are a log scheme S' over A satisfying the equivalent conditions (i)–(iv) in 1.2.5, a morphism $f: S \rightarrow S'$ of log schemes over A , a generalized Drinfeld module $((\mathcal{L}', \phi'), \iota')$ over (S', U) of rank d with level N structure in the sense of 1.2.7, where U denotes the dense open set of S' at which the log structure of S' is trivial, and an isomorphism between $((\mathcal{L}, \phi), \iota)$ and the pullback of $((\mathcal{L}', \phi'), \iota')$ (here ι' is regarded as a map $(\frac{1}{N}A/A)^d \rightarrow \overline{\mathcal{L}'}$ (1.2.7)) under f .

Note that we are not attempting to define the notion of a log Drinfeld module without reference to a level structure.

Remark 1.2.9.

- (1) If S is a scheme over A with trivial log structure, a log Drinfeld module over S of rank d with level N structure is equivalent to a Drinfeld module over S of rank d with Drinfeld level N structure (see 2.7.3).

- (2) A Drinfeld module is additive and a log structure is multiplicative. Hence it is not very easy to connect these two different notions to define a log Drinfeld module. In our formulation, a log Drinfeld module is like a centaur: It has the body $\overline{\mathcal{L}}$ whose half \mathcal{L} is additive and whose the other half $M_S^{-1}\mathcal{L}^\times$ is multiplicative.

We will prove the following two propositions.

Proposition 1.2.10. *Let S be as in 1.2.8, and let $((\mathcal{L}, \phi), \iota)$ be a log Drinfeld module over S . Let $a, b \in (\frac{1}{N}A/A)^d$. Then locally on S , we have either*

$$\frac{\text{pole}(\iota(a))}{\text{pole}(\iota(b))} \in M_S/\mathcal{O}_S^\times \quad \text{or} \quad \frac{\text{pole}(\iota(b))}{\text{pole}(\iota(a))} \in M_S/\mathcal{O}_S^\times \quad \text{in} \quad M_S^{\text{gp}}/\mathcal{O}_S^\times.$$

See 2.7.6 for the proof.

Proposition 1.2.11. *Let S and $((\mathcal{L}, \phi), \iota)$ be as in 1.2.8. Assume $A = \mathbb{F}_q[T]$. Then:*

(1) *Locally on S , there is an A/NA -basis $(e_0)_{0 \leq i \leq d-1}$ of $(\frac{1}{N}A/A)^d$ satisfying the following condition:*

(1.1) *For each nonnegative integer i with $i \leq d-1$, we have*

$$\frac{\text{pole}(\iota(a))}{\text{pole}(\iota(e_i))} \in M_S/\mathcal{O}_S^\times \subset M_S^{\text{gp}}/\mathcal{O}_S^\times$$

for all $a \in (\frac{1}{N}A/A)^d$ with $a \notin \sum_{j=0}^{i-1} (A/NA)e_j$.

Furthermore, we have the following.

(2) *The values $\text{pole}(\iota(e_i))$ with $0 \leq i \leq d-1$ are independent of the choice of basis $(e_i)_i$ in (1.1), and $\text{pole}(\iota(e_0)) = 1$ for every choice of $(e_i)_i$.*

See 3.4.2 for the proof.

1.3 Main results

1.3.1. In the rest of this introduction, we suppose that $A = \mathbb{F}_q[T]$.

We fix N and d as in Section 1.2. In the case N has at least two prime divisors (resp. N has only one prime divisor), let \mathcal{C}_{\log} be the category of schemes over A (resp. $A[\frac{1}{N}]$) with saturated log structures.

1.3.2. Let

$$\overline{\mathfrak{M}}_N^d : \mathcal{C}_{\log} \rightarrow (\text{Sets})$$

be the contravariant functor for which $\overline{\mathfrak{M}}_N^d(S)$ is the set of all isomorphism classes of log Drinfeld modules over S of rank d with level N structure.

1.3.3. Let \mathcal{M}_N^d be as in 1.1.2. We endow \mathcal{M}_N^d with the trivial log structure. Viewed as a functor, $\mathcal{M}_N^d: \mathcal{C}_{\log} \rightarrow (\text{Sets})$ sends an object S to the set $\text{Mor}(S, \mathcal{M}_N^d)$ which consists of all isomorphism classes of Drinfeld modules over the underlying scheme of S over A of rank d endowed with Drinfeld level N structure. So by 1.2.9(1), we have an embedding of functors $\mathcal{M}_N^d \hookrightarrow \overline{\mathfrak{M}}_N^d$.

1.3.4. Consider the cone

$$C_d := \{(s_1, \dots, s_{d-1}) \in \mathbb{R}^{d-1} \mid 0 \leq s_1 \leq \dots \leq s_{d-1}\}.$$

For a finite rational cone decomposition Σ of C_d , we define the subfunctor $\overline{\mathfrak{M}}_{N,\Sigma}^d$ of $\overline{\mathfrak{M}}_N^d$ which contains \mathcal{M}_N^d as follows.

For an object S of \mathcal{C}_{\log} , the set $\overline{\mathfrak{M}}_{N,\Sigma}^d(S)$ consists of all

$$((\mathcal{L}, \phi), \iota) \in \overline{\mathfrak{M}}_N^d(S)$$

such that locally on S , there exist an A/NA -basis $(e_i)_{0 \leq i \leq d-1}$ of $(\frac{1}{N}A/A)^d$ satisfying (1.1) of 1.2.11 and a cone $\sigma \in \Sigma$ such that

$$\prod_{i=1}^{d-1} \text{pole}(\iota(e_i))^{b(i)} \in M_S/\mathcal{O}_S^\times \subset M_S^{\text{gp}}/\mathcal{O}_S^\times$$

for all $b = (b(i))_i \in \mathbb{Z}^{d-1}$ for which $\sum_{i=1}^{d-1} b(i)s_i \geq 0$ for all $(s_i)_i \in \sigma$.

The following theorem is the main result of this paper.

Theorem 1.3.5.

- (1) For every finite rational cone decomposition Σ of C_d , the functor $\overline{\mathfrak{M}}_{N,\Sigma}^d$ is represented by an object $\overline{\mathcal{M}}_{N,\Sigma}^d$ of \mathcal{C}_{\log} . This log scheme $\overline{\mathcal{M}}_{N,\Sigma}^d$ is log regular, and $\overline{\mathcal{M}}_{N,\Sigma}^d \otimes_A A[\frac{1}{N}]$ is log smooth over $A[\frac{1}{N}]$. The underlying scheme of $\overline{\mathcal{M}}_{N,\Sigma}^d$ is proper over A (resp. $A[\frac{1}{N}]$) if N has at least two prime divisors (resp. only one prime divisor).
- (2) If Σ' is a finite rational subdivision of Σ , the morphism $\overline{\mathcal{M}}_{N,\Sigma'}^d \rightarrow \overline{\mathcal{M}}_{N,\Sigma}^d$ is proper, birational, and log étale.
- (3) For a given finite rational cone decomposition Σ of C_d , there is a finite rational subdivision Σ' of Σ such that the underlying scheme of $\overline{\mathcal{M}}_{N,\Sigma'}^d$ is regular, the boundary $\overline{\mathcal{M}}_{N,\Sigma'}^d \setminus \mathcal{M}_N^d$ is a divisor with normal crossings, the underlying scheme of $\overline{\mathcal{M}}_{N,\Sigma'}^d \otimes_A A[\frac{1}{N}]$ is smooth over $A[\frac{1}{N}]$, and $\overline{\mathcal{M}}_{N,\Sigma'}^d \otimes_A A[\frac{1}{N}] \setminus \mathcal{M}_N^d \otimes_A A[\frac{1}{N}]$ is a relative normal crossing divisor over $A[\frac{1}{N}]$.

We call the space $\overline{\mathcal{M}}_{N,\Sigma}^d$ in (1) a toroidal compactification of \mathcal{M}_N^d . The proof of Theorem 1.3.5 is given in Section 5.

Our theory of cone decompositions bases on the decomposition of the Bruhat-Tits building into simplices which is reviewed in Section 2.8, as is explained in Section 3.1.

A relation between the shapes of cones in Σ and local shapes of toroidal compactifications is explained in the following Theorem 1.3.6 (3).

Theorem 1.3.6. *There is a special finite rational cone decomposition Σ_k of C_d for each $k \geq 1$ such that if k is the degree of the polynomial $N \in A = \mathbb{F}_q[T]$, we have the following (1)–(3):*

(1) $\overline{\mathfrak{M}}_N^r = \overline{\mathfrak{M}}_{N, \Sigma_k}^r$.

(2) *For a finite rational cone decomposition Σ of C_d , we have $\overline{\mathcal{M}}_{N, \Sigma}^r = \overline{\mathcal{M}}_{N, \Sigma * \Sigma_k}^r$, where $*$ means the join.*

(3) *Let Σ be a finite rational subdivision of Σ_k . Then there are schemes U_σ for $\sigma \in \Sigma$, an étale surjective morphism $\coprod_{\sigma \in \Sigma} U_\sigma \rightarrow \overline{\mathcal{M}}_{N, \Sigma}^d$, smooth morphisms $U_\sigma \rightarrow \text{toric}_{\mathbb{F}_p}(\sigma)$ over \mathbb{F}_p such that the inverse image of \mathcal{M}_N^d in U_σ coincides with the inverse image of the torus part $\mathbf{G}_{m, \mathbb{F}_p}^{d-1}$ of the toric variety $\text{toric}_{\mathbb{F}_p}(\sigma)$ over \mathbb{F}_p , and étale morphisms $U_\sigma \otimes_A A[\frac{1}{N}] \rightarrow \text{toric}_{A[\frac{1}{N}]}(\sigma)$ over $A[\frac{1}{N}]$ such that the inverse image of $\mathcal{M}_N^r \otimes_A A[\frac{1}{N}]$ in $U_\sigma \otimes_A A[\frac{1}{N}]$ coincides with the inverse image of the torus part $\mathbf{G}_{m, A[\frac{1}{N}]}^{d-1}$ of $\text{toric}_{A[\frac{1}{N}]}(\sigma)$.*

See Section 5 for the proof. The part (3) of Theorem 1.3.5 follows from the part (3) of Theorem 1.3.6 by the fact that there is a finite subdivision Σ' of Σ such that $\text{toric}_{\mathbb{F}_p}(\sigma)$ is smooth over \mathbb{F}_p for all $\sigma \in \Sigma'$.

The following Theorem 1.3.7 shows that log Drinfeld modules over an object S of \mathcal{C}_{\log} are simply understood in the case S is as in 1.2.4.

Theorem 1.3.7. *Let S be an object of \mathcal{C}_{\log} such that the underlying scheme of S is normal and such that the log structure of S is associated to a dense open subset U of S as in 1.2.4. Then the following two notions are equivalent.*

(i) *A log Drinfeld module $((\mathcal{L}, \phi), \iota)$ over S of rank d with level N structure.*

(ii) *A generalized Drinfeld module $((\mathcal{L}, \phi), \iota)$ over (S, U) of rank d with level N structure in the sense of 1.2.7 such that for each $a, b \in (\frac{1}{N}A/A)^d$, locally on S , we have either $\text{pole}(\iota(a)) \text{pole}(\iota(b))^{-1} \in M_S/\mathcal{O}_S^\times$ or $\text{pole}(\iota(b)) \text{pole}(\iota(a))^{-1} \in M_S/\mathcal{O}_S^\times$ in $M_S^{\text{gp}}/\mathcal{O}_S^\times$.*

We obtain (ii) from (i) by restricting ι of (i) to U , and conversely, (i) from (ii) by taking the unique extension of ι in (ii) to a map to $\overline{\mathcal{L}}$ (1.2.7).

This will be proved in Section 5.

1.4 Comparison with abelian varieties

We describe that our constructions of toroidal compactifications are similar to the constructions of the toroidal compactifications of the moduli spaces of abelian varieties in [3, 6] but that there are differences. In particular, we explain the Tate uniformizations of Drinfeld modules on adic spaces, and a theory of iterated Tate uniformizations on formal schemes that we use.

1.4.1. The most difficult part of this paper is to prove that our toroidal compactification is log regular (that is, it has only toric singularities) worthy of the name.

Consider the theory of compactification of the moduli space of elliptic curves with level N structure. The formal completion of the compactified moduli space at the boundary is isomorphic to $\mathrm{Spf}(\mathbb{Z}[\zeta_N][[q^{1/N}]])$, and from this we can deduce that the compactified moduli space is regular and its tensor product with $\mathbb{Z}[\frac{1}{N}]$ is smooth over $\mathbb{Z}[\frac{1}{N}]$. Here q is the q -invariant which appears in the formal moduli theory. The universal degenerate elliptic curve over $\mathrm{Spf}(\mathbb{Z}[\zeta_N][[q^{1/N}]])$ is expressed as the quotient of the multiplicative group by a \mathbb{Z} -lattice of rank 1, which is the theory of Tate uniformization in (1) below, and q is the \mathbb{Z} -basis of this \mathbb{Z} -lattice.

Similarly, to prove the log regularity and the log smoothness of our toroidal compactification, we consider the formal completion of the compactification at the boundary, which is a moduli space of degeneration of Drinfeld modules over formal schemes.

- (1) In the theory of degeneration of abelian varieties over formal schemes, an important thing is an analytic presentation $X = Y/\Lambda$ of an abelian variety X with degeneration as a quotient of a semi-abelian variety Y by a \mathbb{Z} -lattice Λ . This is due to Tate, Raynaud, Mumford, and Faltings-Chai. This Λ is useful to understand the formal completion of the compactified moduli space.
- (2) In the theory of degeneration of Drinfeld modules over formal schemes, in Section 4.2 of this paper, we will obtain the Tate uniformization $X = Y/\Lambda$ of a Drinfeld module with degeneration as a quotient of a Drinfeld module Y by a certain Λ , generalizing the result of Drinfeld over complete discrete valuation fields. Though the result of Drinfeld was useful to understand the compactified moduli space of Drinfeld modules of rank 2, this generalized Tate uniformization is not so useful in general because this Λ , which is a locally constant sheaf of A -lattices on an adic space, has too big monodromy action in the case of rank > 2 and not merely an A -lattice. What is useful is a theory of iterated Tate uniformizations, which expresses X as an iterated quotient of Y . That is, we obtain X from Y via $Y_0 = Y$, $Y_{i+1} = Y_i/\Lambda_i$ ($0 \leq i \leq n-1$) and $X = Y_n$, where Y_i ($1 \leq i \leq n$) are generalized Drinfeld modules and Λ_i ($0 \leq i \leq n-1$) are A -lattices in Y_i of rank 1. These Λ_i are useful to understand the compactified moduli space. See Section 4.5.

The difference between (1) and (2) is explained more in 4.2.20.

1.4.2. The difference between the above (1) and (2) is related to the following difference of local monodromy over a valuation ring.

Let \mathcal{V} be a strictly Henselian valuation ring (which need not be of height one) with field of fractions K . For an abelian variety over K , the local monodromy satisfies $(\sigma_1 - 1)(\sigma_2 - 1) = 0$ after taking a finite extension of K . On the other hand, for a Drinfeld module over K , the local monodromy satisfies $(\sigma_1 - 1)(\sigma_2 - 1) \dots (\sigma_n - 1) = 0$ for some n after a finite extension of K , but we cannot always take $n = 2$. This is discussed in Section 2.4.

1.4.3. In [17], the notion of a log abelian variety is defined, and it is shown that the toroidal compactifications ([3], [6]) of moduli spaces of abelian varieties are understood as the moduli spaces of log abelian varieties. This notion is similar to the notion of a log Drinfeld module in this paper. For a log scheme S , a log abelian variety \mathcal{A} over S is a contravariant functor from the category of fs log schemes over S to the category of abelian groups satisfying certain conditions and which has a subgroup functor represented by a semi-abelian scheme \mathcal{G} over S . The inclusion $\mathcal{G} \subset \mathcal{A}$ is similar to $\mathcal{L} \subset \overline{\mathcal{L}}$. One difference is that $\overline{\mathcal{L}}$ does not have a group structure, though \mathcal{A} does.

1.5 Plan of this paper, acknowledgements

1.5.1. The plan of this paper is as follows. Section 2 contains generalities on Drinfeld modules, generalized Drinfeld modules and log Drinfeld modules. In Section 3, we discuss cone decompositions related to Bruhat-Tits buildings and Drinfeld exponential maps. In Section 4, we discuss Tate uniformizations, and we discuss formal moduli by using iterated Tate uniformizations. In Section 5, we prove our main theorem.

In Section 2 and Sections 4.1, 4.2 and 4.3, we consider general A . In the rest of the paper (that is, Sections 3, Sections 4.4, 4.5, and Section 5), we restrict to the case $A = \mathbb{F}_q[T]$. In a forthcoming paper, Sections 3–5 will be extended to general A .

1.5.2. The reasons why we divide this work into two papers, Part I and Part II, are as follows. First, in the theory for general A , we use reduction to the case $A = \mathbb{F}_q[T]$ in several key points. Hence, it is better to treat the case $A = \mathbb{F}_q[T]$ first. Second, the case $A = \mathbb{F}_q[T]$ is especially simple and the ideas can be seen clearly (for general A , the description of the theory becomes very involved in parts, and the ideas are hidden behind complicated definitions).

1.5.3. In this paper, we use ideas of Richard Pink and Kazuhiro Fujiwara described in the paper [24] of Pink. In [24], Pink uses torsion points of Drinfeld modules for his theory of toroidal compactifications of moduli spaces of Drinfeld modules. Our use of torsion points in the above Section 1.3 is similar to it. The importance of the iterated Tate uniformization appears in Pink [24] at the place where he introduces the study of Fujiwara. The log regularity and the log smoothness of the toroidal compactifications are not discussed in [24].

Though details are not written in [24], we believe that the ideas of Pink and Fujiwara in [24] were enough to have the theory of toroidal compactifications.

1.5.4. The second author wishes to express his thanks to Sampei Usui, Chikara Nakayama, and Takeshi Kajiwara that the joint works with them on partial toroidal compactifications of period domains and on log abelian varieties ([22], [21], [17]) were very helpful to find the right ways in this work.

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2 Generalized and log Drinfeld modules

Let F be a function field in one variable with total constant finite field \mathbb{F}_q . Fix a place ∞ of F . Let A be the ring of all elements of F which are integral outside ∞ . Let F_∞ be the local field of F at ∞ , and let $|\cdot|$ be the standard absolute value of F_∞ .

2.1 Basic things

We review the definitions of a Drinfeld module [5] and of a generalized Drinfeld module (in the terminology of [25]).

Let S denote a scheme over A .

2.1.1. A *generalized Drinfeld module* over S is a pair (\mathcal{L}, ϕ) , where \mathcal{L} is a line bundle over S and ϕ is an \mathbb{F}_q -linear action of the ring A on the (additive) group scheme \mathcal{L} over S satisfying the following conditions. Locally on S , take a basis δ of \mathcal{L} , and write

$$\phi(a)(z\delta) = \sum_{n=0}^{\infty} c(\delta, a, n) z^n \delta$$

($a \in A, z \in \mathcal{O}_S, c(\delta, a, n) \in \mathcal{O}_S$ independent of z and zero for sufficiently large n). Then

- (i) $c(\delta, a, 1) = a$ for all $a \in A$, and
- (ii) for each $s \in S$, there are $a \in A$ and $n \geq 2$ such that $c(\delta, a, n) \in \mathcal{O}_{S,s}^\times$.

A generalized Drinfeld module (\mathcal{L}, ϕ) will frequently be written simply as ϕ . Its local coefficients $c(\delta, a, n)$ for a given δ will also be written $c(a, n)$. By definition, we have

$$c(u\delta, a, n) = c(\delta, a, n) u^{n-1}$$

for $u \in \mathcal{O}_S^\times$. If $\mathcal{O}_S \cong \mathcal{L}$, we will often identify \mathcal{L} with \mathcal{O}_S by a choice of δ and omit the notation for δ in the formula for ϕ .

A homomorphism of generalized Drinfeld modules is a homomorphism of line bundles which is compatible with the given actions of A .

2.1.2. For a generalized Drinfeld module ϕ over S , we have $\phi(a)(z) = az$ for all $a \in \mathbb{F}_q$. From this, it follows that $c(a, n) = 0$ for all $a \in A$ unless n is a power q . That is, locally we have

$$\phi(a)(z) = \sum_{i=0}^{\infty} a_i \tau^i(z)$$

for some $a_i \in \mathcal{O}_S$, where τ is the homomorphism $z \mapsto z^q$.

2.1.3. For a generalized Drinfeld module ϕ over S and $s \in S$, there is an integer $r(s) \geq 1$ such that for each $a \in A \setminus \{0\}$, the coefficient $c(a, |a|^{r(s)})$ is invertible at s and $c(a, n)$ is not invertible at s if $n > |a|^{r(s)}$. This is quickly reduced to the case that $S = \text{Spec}(k)$ for a field k over A and [5, Prop. 2.1]. That $r(s)$ is an integer is [5, Cor. to Prop. 2.2] (see also [10, Prop. 4.5.3]).

2.1.4. A generalized Drinfeld module over S is called a *Drinfeld module* over S of rank d if it satisfies

- (i) $r(s) = d$ for all $s \in S$, where $r(s)$ is as in 2.1.3, and
- (ii) $c(a, n) = 0$ if $a \in A$ and $n > |a|^d$.

Remark 2.1.5. The definition of an elliptic module over S of rank d and the definition of a standard elliptic module over S of rank d are given in (B) of [5, Section 5]. Actually, the above definition of a Drinfeld module over S of rank d is the same as the definition of a standard elliptic module over S of rank d . As is explained in [5, Section 5], the category of Drinfeld modules (i.e., standard elliptic modules) over S of rank d is equivalent to the category of elliptic modules over S of rank d .

2.1.6. In the case that S is an integral scheme, by a generalized Drinfeld module over S of *generic rank* d we mean a generalized Drinfeld module over S whose restriction to the generic point of S is a Drinfeld module of rank d .

Note that for a generalized Drinfeld module ϕ over S of generic rank d , we have $r(s) \leq d$ for every $s \in S$, and ϕ is a Drinfeld module if and only if $r(s) = d$ for all $s \in S$.

2.2 Stable reduction theorem

The following proposition and its corollary are generalizations of [5, Prop. 7.1]. The proofs are essentially the same as those given in [5, Section 7].

Let S be an integral scheme over A , and let K be its function field. We require the following lemma.

Lemma 2.2.1. *Suppose that S is normal. Let (\mathcal{L}, ϕ) and (\mathcal{L}', ϕ') be generalized Drinfeld modules over S , and let $f: (\mathcal{L}, \phi)_K \xrightarrow{\sim} (\mathcal{L}', \phi')_K$ be an isomorphism between their pullbacks to $\text{Spec } K$. Then f induces an isomorphism $(\mathcal{L}, \phi) \xrightarrow{\sim} (\mathcal{L}', \phi')$.*

Proof. We may assume that $(\mathcal{L}, \phi)_K = (\mathcal{L}', \phi')_K$ and f is the identity map. It suffices to prove that $\mathcal{L} = \mathcal{L}'$. Let $s \in S$, and let δ and δ' be basis elements of \mathcal{L} and \mathcal{L}' at s , respectively. Write $\delta' = u\delta$ with $u \in K^\times$. Choose $a, a' \in A$ and integers $n, n' \geq 2$ such that $c(\delta, a, n), c(\delta', a', n') \in \mathcal{O}_{S,s}^\times$. Since $c(\delta', a, n) \in \mathcal{O}_{S,s}$, we have $u^{n-1} \in \mathcal{O}_{S,s}$, and similarly $c(\delta, a', n') \in \mathcal{O}_{S,s}$ implies that $u^{1-n'} \in \mathcal{O}_{S,s}$. Since S is normal, we have $u \in \mathcal{O}_{S,s}^\times$. Since this holds for all s , we have $\mathcal{L} = \mathcal{L}'$. \square

Proposition 2.2.2. *Let U be a dense open subset of S , and let ϕ be a generalized Drinfeld module over U . There exist a proper birational morphism $T \rightarrow S$ and a finite separable extension K' of K such that if T' denotes the integral closure of T in K' and U' denotes the inverse image of U in T' , then the pullback of ϕ to U' extends uniquely to a generalized Drinfeld module over T' .*

Proof. Take a finite set of \mathbb{F}_q -algebra generators y_i of A with $1 \leq i \leq k$. Fix a linear basis element δ of the Drinfeld module over K , and define $c(y_i, n) \in K$ using δ . Let Y be the finite set of all pairs (i, n) of integers with $1 \leq i \leq k$ and $n \geq 2$ such that $c(y_i, n) \neq 0$. Let K' be the finite extension of K obtained by adjoining all $(n-1)$ th roots of $c(y_i, n)$ for all $(i, n) \in Y$. Since for every $(i, n) \in Y$, the integer n is a power of q by 2.1.2, the field K' is separable over K .

Take an integer $m \geq 1$ which satisfies $\frac{m}{n-1} \in \mathbb{Z}$ for all $(i, n) \in Y$, and consider the fractional \mathcal{O}_S -ideal I on S generated by the $c(y_i, n)^{m/(n-1)}$. Let $T \rightarrow S$ be the blowup of S by I , and let T' be the integral closure of T in K' . For $(i, n) \in Y$, let $U(i, n)$ be the open part of T on which $c(y_i, n)^{m/(n-1)}$ divides $c(y_j, h)^{m/(h-1)}$ for all $(j, h) \in Y$. Then the $U(i, n)$ form an open covering of T . Let $U'(i, n)$ be the inverse image of $U(i, n)$ in T' .

Let \mathcal{L}' be the line bundle on T' for which $\delta_{i,n} := c(y_i, n)^{-1/(n-1)}\delta$ is a basis on $U'(i, n)$. For $x \in \mathcal{O}_{U'(i,n)}$, we set

$$\phi'(y_j)(x\delta_{i,n}) = y_j x \delta_{i,n} + \sum_h x^h \left(\frac{c(y_j, h)^{1/(h-1)}}{c(y_i, n)^{1/(n-1)}} \right)^{h-1} \delta_{i,n},$$

where h ranges over all integers such that $(j, h) \in Y$. Since

$$\frac{c(y_j, h)^{1/(h-1)}}{c(y_i, n)^{1/(n-1)}} \in \mathcal{O}_{U'(i,n)}$$

by definition, (\mathcal{L}', ϕ') is a generalized Drinfeld module over T' . Over K' , this Drinfeld module ϕ' is the pullback of ϕ on K . Since T' is normal, Lemma 2.2.1 tells us that ϕ' is the unique extension to T' of the pullback of ϕ to U' . \square

Taking S to be the spectrum of a valuation ring in Proposition 2.2.2, we have the following corollary (as the valuative criterion for properness forces $T = S$).

Corollary 2.2.3. *Let \mathcal{V} be a valuation ring with field of fractions K , and let ϕ be a Drinfeld module over K . Then there is a finite extension K' of K such that the pullback of ϕ to K' comes from a generalized Drinfeld module over the integral closure of \mathcal{V} in K' .*

2.3 Generalized Drinfeld modules over complete rings

We give complements Prop. 2.3.1 and Prop. 2.3.5 to [5, Section 7].

Proposition 2.3.5 says that the complete discrete valuation ring in [5, Prop. 7.2] is generalized to a complete valuation ring of height one. Proposition 2.3.1 says that a part of [5, Prop. 7.2]

is generalized to commutative rings which are I -adically complete for an ideal I . The proofs of 2.3.1 and 2.3.5 are essentially the same as the arguments given in [5, Section 7].

Proposition 2.3.1. *Let R be a commutative ring over A , and let I be an ideal of R such that $R \xrightarrow{\sim} \varprojlim_i R/I^i$. Let ϕ be a generalized Drinfeld module over R with trivial line bundle, and suppose it reduces to a Drinfeld module over R/I of rank r . Then there is a unique pair (ψ, e) , where ψ is a Drinfeld module over R of rank r with trivial line bundle and $\psi \equiv \phi \pmod{I}$, and where $e \in R[[z]]$ is such that $e \equiv z \pmod{I}$, the z^n -coefficient of e is zero unless n is a power of q , the reduction modulo I^n of e is a polynomial for every n , and $e \circ \psi(a) = \phi(a) \circ e$ for all $a \in A$.*

Proof. As in the first five lines of part (2) of the proof of [5, Prop. 7.2], the result follows from [5, Prop. 5.2] (taking $B = R/I^n$ therein). More details are given in the argument of [25, Prop. 3.4]. Very roughly, [5, Prop. 5.2] is used to find the existence of a unique e and $\psi(b)$ such that $e \circ \psi(b) = \phi(b) \circ e$, given a nonconstant $b \in A$. By the commutativity of $e\phi(a)e^{-1}$ and $e\phi(b)e^{-1}$ for every $a \in A$, we can use [5, Prop. 5.1] to see that the same e works for $\phi(a)$. \square

Remark 2.3.2. There is a natural generalization of 2.3.1 to the situation where we do not assume that the line bundle \mathcal{L} of ϕ is trivial. We have still a correspondence $\phi \mapsto (\psi, e)$ where the line bundle of ψ is \mathcal{L} and $e = \sum_{i=0}^{\infty} c_i \tau_i$ where $c_i \in R$ and τ_i is a semi-linear map $\mathcal{L} \rightarrow \mathcal{L}$ with respect to the ring homomorphism $R \rightarrow R$; $x \mapsto x^{q^i}$.

In the rest of this Section 2.3, Drinfeld (resp. generalized Drinfeld) modules are assumed to have trivialized line bundles.

2.3.3. Let \mathcal{V} be a valuation ring over A with maximal ideal \mathfrak{m} , and let K be the field of fractions of \mathcal{V} . Fix an algebraic closure \bar{K} of K , let $K^{\text{sep}} \subset \bar{K}$ be the separable closure of K , and set $G_K = \text{Gal}(K^{\text{sep}}/K)$. Let v_K be an additive valuation of K associated to \mathcal{V} . We suppose that

- \mathcal{V} is of height one, i.e., the value group of v_K is isomorphic as an ordered group to a subgroup of \mathbb{R} , and
- \mathcal{V} is complete, i.e., for every nonzero $a \in \mathfrak{m}$, the canonical map $\mathcal{V} \rightarrow \varprojlim_i \mathcal{V}/a^i \mathcal{V}$ is an isomorphism.

Let $v_{\bar{K}}$ denote the unique extension of v_K to an additive valuation of \bar{K} .

For a Drinfeld module ψ over K , by a $\psi(A)$ -lattice in K^{sep} , we mean a projective G_K -stable $\psi(A)$ -submodule Λ of finite type of K^{sep} such that $\{x \in \Lambda \mid v_{\bar{K}}(x) \geq c\}$ is finite for all $c \in \mathbb{R}$.

We say that a Drinfeld module ψ over K has potentially good reduction if it comes from a Drinfeld module over the integral closure of \mathcal{V} in some finite extension of K .

The following lemma will be useful later.

Lemma 2.3.4. *Let the notation be as in 2.3.3. Let ψ be a Drinfeld module over \mathcal{V} of rank r , and let Λ be a $\psi(A)$ -lattice in K^{sep} . Then $v_{\bar{K}}(\lambda) < 0$ for all $\lambda \in \Lambda \setminus \{0\}$.*

Proof. If $\lambda \in \Lambda$ is nonzero and $v_{\bar{K}}(\lambda) \geq 0$, then $v_{\bar{K}}(\psi(a)\lambda) \geq 0$ for all $a \in A$, and this contradicts the finiteness of $\{\lambda \in \Lambda \mid v_{\bar{K}}(\lambda) \geq 0\}$. \square

Proposition 2.3.5. *Let the notation be as in 2.3.3.*

- (1) *Generalized Drinfeld modules of generic rank d over \mathcal{V} are in one-to-one correspondence with pairs (ψ, Λ) consisting of a Drinfeld module ψ over \mathcal{V} of rank $r \leq d$ and a $\psi(A)$ -lattice Λ in \bar{K} of rank $d - r$.*
- (2) *Drinfeld modules of rank d over K are in one-to-one correspondence with pairs (ψ, Λ) consisting of a Drinfeld module ψ over K of rank $r \leq d$ with potentially good reduction and a $\psi(A)$ -lattice Λ in K^{sep} of rank $d - r$.*

Proof. The proof is similar to that of [5, Prop. 7.2]. Given a generalized Drinfeld module ϕ over \mathcal{V} of rank d , its reduction modulo \mathfrak{m} is a Drinfeld module of some rank $r \leq d$. Let t be a nonzero element in \mathfrak{m} . Then $\psi \bmod t$ is also a Drinfeld module of rank r . Moreover, as \mathcal{V} is complete for the ideal generated by t , Proposition 2.3.1 and Remark 2.3.2 provide a unique pair (ψ, e) with ψ a Drinfeld module over \mathcal{V} of rank r and, after trivializing the line bundle of ϕ , with $e \in \mathcal{V}[[z]]$ such that $e \equiv z \bmod t$ and $e \circ \psi(a) = \phi(a) \circ e$ for all $a \in A$. The lattice Λ is then taken to be the kernel of e in K^{sep} . \square

2.4 Local monodromy over valuation rings

2.4.1. Let \mathcal{V} be a strictly Henselian valuation ring over A with maximal ideal \mathfrak{m} and field of fractions K . Let K^{sep} denote a separable closure of K . Let ϕ be a generalized Drinfeld module over \mathcal{V} with generic rank d .

For a place $v \neq \infty$ of F that does not correspond to the kernel of $A \rightarrow \mathcal{V}/\mathfrak{m}$, let F_v denote the completion at v and O_v its valuation ring. Consider the v -adic Tate module

$$T_v(\phi) = \text{Hom}_A(F_v/O_v, \phi\{v\})$$

of ϕ , where $\phi\{v\}$ is the v -primary torsion of ϕ in K^{sep} . Then $T_v(\phi)$ is a free O_v -module of rank d with a continuous action of G_K . Let $V_v(\phi) = F_v \otimes_{O_v} T_v(\phi)$.

For each prime ideal \mathfrak{p} of \mathcal{V} , let $r(\mathfrak{p})$ denote the rank of the Drinfeld module over the residue field $\mathcal{V}_{\mathfrak{p}}/\mathfrak{p}\mathcal{V}_{\mathfrak{p}}$ of \mathfrak{p} obtained from ϕ . Let

$$0 = d(-1) < d(0) < \cdots < d(m) = d$$

be such that

$$\{d(i) \mid 0 \leq i \leq m\} = \{r(\mathfrak{p}) \mid \mathfrak{p} \in \text{Spec}(\mathcal{V})\}.$$

Theorem 2.4.2. *The action of G_K on $V_v(\phi)$ is quasi-unipotent. More precisely, the following statements hold.*

- (1) For a sufficiently small open subgroup H of G_K , there are G_K -stable F_v -subspaces V_i of $V_v(\phi)$ of dimension $d(i)$ for $-1 \leq i \leq m$ such that

$$0 = V_{-1} \subset V_0 \subset \cdots \subset V_m = V_v(\phi)$$

and H acts trivially on the graded subquotients V_i/V_{i-1} with $0 \leq i \leq m$.

- (2) Assume furthermore that the value group Γ of \mathcal{V} has the property that $\Gamma \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ for $p = \text{char}(\mathbb{F}_q)$ is a finitely generated $\mathbb{Z}_{(p)}$ -module. Then $V_{i-1} = I_H V_i$ for all $0 \leq i \leq m$, where I_H denotes the augmentation ideal in the group ring $F_v[H]$.

Proof. We fix a trivialization of the line bundle of ϕ . For each positive integer $i \leq m$, let \mathfrak{P}_i be the union of all prime ideals \mathfrak{p} of \mathcal{V} such that $r(\mathfrak{p}) = d(i)$, and let \mathfrak{Q}_i be the intersection of all prime ideals \mathfrak{q} of \mathcal{V} such that $r(\mathfrak{q}) = d(i-1)$. Since the set of ideals of a valuation ring is totally ordered, \mathfrak{P}_i and \mathfrak{Q}_i are prime ideals of \mathcal{V} with $r(\mathfrak{P}_i) = d(i)$, $r(\mathfrak{Q}_i) = d(i-1)$, and $\mathfrak{P}_i \subsetneq \mathfrak{Q}_i$. The image of the local ring $\mathcal{V}_{\mathfrak{Q}_i}$ in the residue field $\kappa_i = \mathcal{V}/\mathfrak{P}_i$ is a valuation ring of height one as it has a unique nonzero prime ideal. Let \mathcal{V}_i be the completion of this image. Let ϕ_i be the generalized Drinfeld module over \mathcal{V}_i induced by ϕ . Let (ψ_i, Λ_i) be the corresponding pair as in 2.3.5(1). We also let $\mathcal{V}_0 = \mathcal{V}$, $\phi_0 = \phi$, and $\mathfrak{P}_0 = \mathfrak{m}$.

For each nonnegative integer $i \leq m$, let ϕ'_i be the Drinfeld module of rank $d(i)$ over κ_i given by ϕ_i . We have a G_K -stable F_v -subspace V_i of $V_v(\phi)$ as follows. Let $\phi\{v\}_i$ be the subgroup of $\phi\{v\}$ consisting of all elements which belong to the integral closure of the local ring $\mathcal{V}_{\mathfrak{P}_i}$ in K^{sep} . Let

$$T_i = \text{Hom}_A(F_v/O_v, \phi\{v\}_i)$$

and $V_i = F_v \otimes_{O_v} T_i$. The map $\phi\{v\}_i \rightarrow \phi'_i\{v\}$ is an isomorphism, and hence we have isomorphisms $T_i \xrightarrow{\sim} T_v(\phi'_i)$ and $V_i \xrightarrow{\sim} V_v(\phi'_i)$. It follows that $\dim_{F_v}(V_i) = d(i)$.

For each positive integer $i \leq m$, we have an exact sequence

$$0 \rightarrow T_{i-1} \rightarrow T_i \rightarrow O_v \otimes_A \Lambda_i \rightarrow 0.$$

By construction, the G_K -action on Λ_i factors through the absolute Galois group of κ_i . Since \mathcal{V} is Henselian, the local ring $\mathcal{V}_{\mathfrak{Q}_i}$ and its image in κ_i are also Henselian. Hence the absolute Galois group of κ_i and that of the field of fractions of \mathcal{V}_i are isomorphic. Hence an open subgroup of G_K acts trivially on Λ_i . This proves (1) and therefore the quasi-unipotence of the G_K -action.

We prove (2). By replacing ϕ by ϕ_i ($1 \leq i \leq m$) and replacing \mathcal{V} by \mathcal{V}_i , we may assume that \mathcal{V} is of complete of height one, \mathfrak{P}_i is the zero ideal, \mathfrak{Q}_i is the maximal ideal of \mathcal{V} , and $i = 1$. By replacing \mathcal{V} by its integral closure in some finite separable extension of K , we may assume that G_K acts on Λ_1 trivially. Write Λ_1 as Λ and ψ_1 as ψ .

Take an element f of A such that $(f) = v^a$ for some $a \geq 1$. We identify $T_v(\phi)$ with $\varprojlim_n \phi[f^n]$, where $\phi[f^{n+1}] \rightarrow \phi[f^n]$ is $\phi(f)$. Let Ω be the set of all families $(x_j)_{j \geq 0}$ such that $x_j \in K^{\text{sep}}$,

$x_0 \in \Lambda$, and $\psi(f)x_{j+1} = x_j$ for all $j \geq 0$. Then we have a commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_0 & \longrightarrow & \Omega & \longrightarrow & \Lambda \longrightarrow 0 \\ & & \parallel & & \downarrow f & & \downarrow \\ 0 & \longrightarrow & T_0 & \longrightarrow & T_v(\phi) & \longrightarrow & O_v \otimes_A \Lambda \longrightarrow 0, \end{array}$$

where $f((x_j)_j) = (e(x_j))_j$ for $e \in \mathcal{V}[[z]]$ such that $e \circ \psi(a) = \phi(a) \circ e$ for all $a \in A$.

Take $\lambda \in \Lambda \setminus \{0\}$ and take $\tilde{\lambda} = (\lambda_j)_{j \geq 0} \in \Omega$ such that $\lambda_0 = \lambda$. Regard $\tilde{\lambda}$ as an element of $T_v(\phi)$ by the above commutative diagram. Let c be the order of the torsion part of $(\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \Gamma) / \mathbb{Z}_{(p)} v_K(\lambda)$. As $v_K(\lambda) < 0$ by Lemma 2.3.4, we have

$$\lambda = \psi(f^j)\lambda_j = u\lambda_j^{|f|^{d(0)j}},$$

where u is a unit of \mathcal{V} . The ramification index of $K(\lambda_j)/K$ is then $|f|^{d(0)j}c^{-1}$, so

$$[K(\lambda_j) : K] \geq |f|^{d(0)j}c^{-1}.$$

Since the degree of the latter extension is the number of distinct conjugates of λ_j , the image of the function $G_K \rightarrow T_0/\phi(f^j)T_0$ given by $g \mapsto (g-1)\lambda_j$ has order at least $|f|^{d(0)j}c^{-1}$. Since $T_0/\phi(f^j)T_0$ has order $|f|^{d(0)j}$, this image has index $\leq c$. Since j is arbitrary, the image of the map $G_K \rightarrow T_0$ given by $g \mapsto (g-1)\tilde{\lambda}$ generates an O_v -submodule of T_0 also of index $\leq c$. In particular, the image of the latter map generates V_0 over F_v , proving (2). \square

We consider a result for abelian varieties corresponding to Theorem 2.4.2.

Proposition 2.4.3. *Let \mathcal{V} be a strictly Henselian valuation ring with field of fractions K . Let B be an abelian variety over K , and let ℓ be prime number which does not coincide with the characteristic of \mathcal{V}/\mathfrak{m} . Then there is an open subgroup H of G_K such that $(\sigma_1 - 1)(\sigma_2 - 1) = 0$ on $T_\ell(B)$ for all $\sigma_1, \sigma_2 \in H$.*

This should be very well-known, but we give a proof.

Proof. The abelian variety B over K (with a polarization and level structure) gives a morphism from $\text{Spec}(K)$ to the moduli space. By the valuative criterion, this morphism extends to a morphism from $\text{Spec}(\mathcal{V})$ to a toroidal compactification of the moduli space. At each point of this toroidal compactification, the universal abelian variety on the moduli space has local monodromy that is unipotent of length two [6]. \square

2.4.4. From 2.4.2(2), we see that the analogue of 2.4.3 for Drinfeld modules is not true. For example, take $A = \mathbb{F}_q[[T]]$, let k be a separably closed field over A , and let

$$\mathcal{V} = k[[t_1]] + t_2k((t_1))[[t_2]] \subset k((t_1, t_2)).$$

Then \mathcal{V} is a strictly Henselian valuation ring and the value group of \mathcal{V} is isomorphic to \mathbb{Z}^2 with the lexicographic order. Consider the generalized Drinfeld module over \mathcal{V} defined by

$$\phi(T)(z) = Tz + z^q + t_1 z^{q^2} + t_2 z^{q^3}.$$

Let $\mathfrak{p}_1 = (t_1) \supsetneq \mathfrak{p}_2 = (t_2) \supsetneq \mathfrak{p}_3 = 0$ be the prime ideals of \mathcal{V} . Then the Drinfeld module over the residue field of \mathfrak{p}_i induced by ϕ has rank i for $i = 1, 2, 3$. Hence 2.4.2(2) shows that if v is a place of A such that $v \neq \infty$ and such that $A \rightarrow k$ does not factor through the residue field of v , there is no open subgroup of G_K such that $(\sigma_1 - 1)(\sigma_2 - 1) = 0$ on $V_v(\phi)$ for all $\sigma_1, \sigma_2 \in H$.

2.5 Log geometry and toric geometry

2.5.1. Let \mathcal{S} be a site with a sheaf of commutative rings \mathcal{O} . A log structure M on \mathcal{S} is a sheaf of commutative monoids endowed with a homomorphism $\alpha: M \rightarrow \mathcal{O}$ of sheaves of monoids for the multiplicative structure of \mathcal{O} such that the map $\alpha^{-1}(\mathcal{O}^\times) \xrightarrow{\alpha} \mathcal{O}^\times$ is an isomorphism. Hence, \mathcal{O}^\times is regarded as a subgroup sheaf of M via α . We say the log structure is trivial if $M = \mathcal{O}^\times$. The semigroup law of the log structure M is written multiplicatively.

By a saturated monoid, we mean a commutative monoid P which satisfies the following conditions (i) and (ii).

(i) In P , if $ab = ac$, then $b = c$. That is, for the group completion $P^{\text{gp}} = \{ab^{-1} \mid a, b \in P\}$ of P , the canonical homomorphism $P \rightarrow P^{\text{gp}}$ is injective.

(ii) If $a \in P^{\text{gp}}$ and if $a^n \in P \subset P^{\text{gp}}$ for some $n \geq 1$, then $a \in P$.

A finitely generated saturated monoid is called an fs monoid.

We say a log structure M is saturated if it is a sheaf of saturated monoids.

A log structure M is called an fs log structure if locally on \mathcal{S} , there is an fs monoid P and a homomorphism $\alpha: P \rightarrow \mathcal{O}$ (called a chart of M) such that M is isomorphic to the pushout of $P \leftarrow \alpha^{-1}(\mathcal{O}^\times) \rightarrow \mathcal{O}^\times$ in the category of sheaves of commutative monoids on \mathcal{S} , which is endowed with the natural homomorphism to \mathcal{O} . An fs log structure is saturated.

Let M be a saturated log structure on \mathcal{S} and let \mathcal{L} be an \mathcal{O} -module on \mathcal{S} which is locally free of rank 1. Then we define a sheaf $\overline{\mathcal{L}}$ on \mathcal{S} as $\mathcal{L} \cup_{\mathcal{L}^\times} (M^{-1} \times^{\mathcal{O}^\times} \mathcal{L}^\times)$. It is regarded as the twist $\mathcal{L}^\times \times^{\mathcal{O}^\times} (\mathcal{O} \cup_{\mathcal{O}^\times} M^{-1})$ of $\mathcal{O} \cup_{\mathcal{O}^\times} M^{-1}$ by the \mathcal{O}^\times -torsor \mathcal{L}^\times .

In this paper, we consider log structures in the following cases.

(a) A log structure on the étale site \mathcal{S} of a scheme S with $\mathcal{O} = \mathcal{O}_S$.

(b) A log structure on the étale site \mathcal{S} of a locally Noetherian formal scheme S with $\mathcal{O} = \mathcal{O}_S$.

(c) A log structure on the site of open sets \mathcal{S} of an adic space S with $\mathcal{O} = \mathcal{O}_S^+$.

(d) A log structure on the étale site \mathcal{S} of an adic space S with $\mathcal{O} = \mathcal{O}_S^+$.

In (c) and (d), recall that an adic space has two sheaves of commutative rings \mathcal{O}_S and $\mathcal{O}_S^+ \subset \mathcal{O}_S$ ([13]). For the étale site of a formal scheme, see [12]. For the étale site of an adic space, see [14].

In the case (a) (resp. (b)), S endowed with a log structure is called a log scheme (resp. a log locally Noetherian formal scheme). A scheme endowed with an fs log structure is called an fs log scheme. The category of fs log schemes has fiber products.

In this paper, we will use saturated log structures. In the applications in later sections in this paper, we will use log structures on adic spaces in (c) and (d) only in the following restricted ways. When we use (c), we will consider only the special log structure $M = \mathcal{O}_S^+ \cap \mathcal{O}_S^\times \subset \mathcal{O}_S$ endowed with the inclusion map $\alpha : M \rightarrow \mathcal{O} = \mathcal{O}_S^+$. When we use (d), we will consider only whether the log structure is trivial or not.

2.5.2. Let S be an fs log scheme.

For a line bundle \mathcal{L} on S , we can understand $\overline{\mathcal{L}}$ as the sheaf of S -morphisms to the projective bundle $\mathbf{P}_S(\mathcal{L} \oplus \mathcal{O}_S) \supset \mathcal{L}$ which is endowed with the log structure as the fiber product of $S \rightarrow S^\circ \leftarrow P$ where S° is the scheme S with the trivial log structure and $P = \mathbf{P}_S(\mathcal{L} \oplus \mathcal{O}_S)$ is endowed with the log structure associated to the Cartier divisor $P \setminus \mathcal{L}$.

A generalization of this to a vector bundle V on S is as follows. Let \overline{V} be the sheaf of morphisms to the projective bundle $P = \mathbf{P}_S(V \oplus \mathcal{O}_S)$ endowed with the log structure defined similarly to above by using the Cartier divisor $P \setminus V$. We have $\overline{V} = V \cup_{V^\times} M_S^{-1} V^\times$ where $V^\times = \{v \in V \mid v \text{ is a part of a base of } V\}$. The authors wonder whether \overline{V} can be used to formulate log shtuka, the shtuka version of log Drinfeld module, for a shtuka is a vector bundle.

We review basic things about toric geometry (the geometric meaning of cone decompositions in toric geometry, etc.) formulated in terms of log structures.

2.5.3. In the rest of this Section 2.5, let $L_{\mathbb{Z}}$ be a finitely generated free \mathbb{Z} -module and let

$$L_{\mathbb{Z}}^* = \text{Hom}_{\mathbb{Z}}(L_{\mathbb{Z}}, \mathbb{Z}).$$

Set $L_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} L_{\mathbb{Z}}$ and $L_{\mathbb{R}}^* = \mathbb{R} \otimes_{\mathbb{Z}} L_{\mathbb{Z}}^*$, and let

$$(\ , \) : L_{\mathbb{R}} \times L_{\mathbb{R}}^* \rightarrow \mathbb{R}$$

be the canonical pairing.

For a finitely generated cone σ in $L_{\mathbb{R}}$, let

$$\sigma^* = \{x \in L_{\mathbb{R}}^* \mid (y, x) \geq 0 \text{ for all } y \in \sigma\}.$$

Then σ^* is a finitely generated cone in $L_{\mathbb{R}}^*$, and we have

$$\sigma = \{y \in L_{\mathbb{R}} \mid (y, x) \geq 0 \text{ for all } x \in \sigma^*\}.$$

Then σ is rational if and only if σ^* is rational.

Let

$$\sigma^\vee := \sigma^* \cap L_{\mathbb{Z}}.$$

Then σ^\vee is an fs monoid.

2.5.4. Let $L_{\mathbb{Z}}$ and $L_{\mathbb{Z}}^*$ be as in 2.5.3.

By a finite rational fan in $L_{\mathbb{R}}$, we mean a finite set of finitely generated rational sharp cones in $L_{\mathbb{R}}$ such that

- (i) if $\sigma \in \Sigma$, all faces of σ belong to Σ ,
- (ii) if $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is a face of σ .

For example, if σ is a finitely generated rational sharp cone in $L_{\mathbb{R}}$, the set $\text{face}(\sigma)$ of all faces of σ is a finite rational fan in $L_{\mathbb{R}}$.

If Σ is a finite rational fan in $L_{\mathbb{R}}$, by a finite rational subdivision of Σ , we mean a finite rational fan Σ' in $L_{\mathbb{R}}$ such that.

- (iii) for each $\tau \in \Sigma'$, there is $\sigma \in \Sigma$ such that $\tau \subset \sigma$,
- (iv) $\bigcup_{\sigma \in \Sigma} \sigma = \bigcup_{\tau \in \Sigma'} \tau$.

In the case $\Sigma = \text{face}(\sigma)$, a finite rational subdivision of Σ is called a finite rational cone decomposition of σ and is called also a finite rational subdivision of σ .

2.5.5. Let Σ be a finite rational fan in $L_{\mathbb{R}}$. Let \mathcal{S} be a site with a sheaf of commutative rings \mathcal{O} and with a saturated log structure M . We define the sheaves $\text{toric}(\Sigma)$ and $[\Sigma]$ on \mathcal{S} by

$$\text{toric}(\Sigma) = \bigcup_{\sigma \in \Sigma} \text{toric}(\sigma) \subset L_{\mathbb{Z}} \otimes_{\mathbb{Z}} M^{\text{gp}} \quad \text{where } \text{toric}(\sigma) := \mathcal{H}om(\sigma^{\vee}, M),$$

$$[\Sigma] = \bigcup_{\sigma \in \Sigma} [\sigma] \subset L_{\mathbb{Z}} \otimes_{\mathbb{Z}} (M^{\text{gp}}/\mathcal{O}^{\times}) \quad \text{where } [\sigma] := \mathcal{H}om(\sigma^{\vee}, M/\mathcal{O}^{\times}),$$

where \bigcup are the unions as sheaves. In the case $\Sigma = \text{face}(\sigma)$, we have $\text{toric}(\Sigma) = \text{toric}(\sigma)$ and $[\Sigma] = [\sigma]$

2.5.6. Let Σ be as in 2.5.5. In the cases (a)–(d) in 2.5.1, let

$$\text{toric}(\Sigma)(S) = \Gamma(\mathcal{S}, \text{toric}(\Sigma)), \quad [\Sigma](S) = \Gamma(\mathcal{S}, [\Sigma]).$$

In the case (a), the functor $S \mapsto \text{toric}(\Sigma)(S)$ is represented by the toric variety $\text{toric}_{\mathbb{Z}}(\Sigma)$ over \mathbb{Z} associated to Σ in the classical toric geometry regarded as a log smooth fs log scheme over \mathbb{Z} : $\text{toric}_{\mathbb{Z}}(\Sigma)$ is the union of its open sets $\text{toric}_{\mathbb{Z}}(\sigma) = \text{Spec}(\mathbb{Z}[\sigma^{\vee}])$ ($\sigma \in \Sigma$) which represent $\text{toric}(\sigma)$ and which are endowed with the fs log structures associated to the chart $\sigma^{\vee} \rightarrow \mathbb{Z}[\sigma^{\vee}]$. In all the cases (a)–(d), $\text{toric}(\Sigma)(S)$ is identified with the set morphisms $(S, \mathcal{O}, M) \rightarrow \text{toric}_{\mathbb{Z}}(\Sigma)$ of locally ringed spaces with log structures. By this and by the fact $\text{toric}_{\mathbb{Z}}(\Sigma) = \bigcup_{\sigma \in \Sigma} \text{toric}_{\mathbb{Z}}(\sigma)$ is an open covering, we have the following 2.5.7.

In the case (a) (resp. (b), resp. (d)), by the weaker topology on \mathcal{S} , we mean we regard the category \mathcal{S} as another site in which we take open coverings of schemes (resp. formal schemes, resp. adic spaces) as coverings. Thus the weaker topology on \mathcal{S} in the case (a) means Zariski topology.

Lemma 2.5.7. *In the case (a) (resp. (b), resp. (d)), the sheaf $\text{toric}(\Sigma)$ on the étale site \mathcal{S} of S , which is the union of $\text{toric}(\sigma)$ ($\sigma \in \Sigma$) as a sheaf of the étale site, is actually the union of $\text{toric}(\sigma)$ as the sheaf for the weaker topology on \mathcal{S} .*

2.5.8. Let Σ be as above and let $S, \mathcal{S}, \mathcal{O}$ and M be as one of the cases (a)–(d) in 2.5.1,

We have

$$L_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}^{\times} \subset \text{toric}(\Sigma) \subset L_{\mathbb{Z}} \otimes_{\mathbb{Z}} M^{\text{gp}},$$

$L_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}^{\times}$ acts on $\text{toric}(\Sigma)$, and

$$\text{toric}(\Sigma)/(L_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}^{\times}) \cong [\Sigma]$$

also as sheaves on \mathcal{S} .

Lemma 2.5.9. *In the case (a) (resp. (b), resp. (d)),*

$$\text{toric}(\Sigma)/(L_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}^{\times}) \cong [\Sigma]$$

as sheaves for the weaker topology on \mathcal{S} .

Proof. The stalks $\mathcal{O}_{S,s}$ (resp. $\mathcal{O}_{S,s}$, resp. $\mathcal{O}_{S,s}^+$) for $s \in S$ are local rings. Hence the exact sequence $0 \rightarrow \mathcal{O}^{\times} \rightarrow M^{\text{gp}} \rightarrow M^{\text{gp}}/\mathcal{O}^{\times} \rightarrow 0$ of sheaves on \mathcal{S} is exact also for the weaker topology because for each object S' of \mathcal{S} , each element of $H^1(S', \mathcal{O}^{\times})$ vanishes locally for the weaker topology. This shows that the map $\text{toric}(\Sigma) \rightarrow [\Sigma]$ is surjective as a map of sheaves for the weaker topology. This proves 2.5.9. \square

Lemma 2.5.10. *In the case (a) (resp. (b), resp. (d)), the sheaf $[\Sigma]$ on the étale site of S , which is the union of $[\sigma]$ ($\sigma \in \Sigma$) as the sheaf on the étale site, is actually the union of $[\sigma]$ as the sheaf for the weaker topology.*

Proof. This follows from 2.5.7 and 2.5.9. \square

Lemma 2.5.11. *Assume we are in one of the cases (a) (resp. (b), resp. (d)) in 2.5.1, let M be a saturated log structure on \mathcal{S} , and let $f, g \in \Gamma(\mathcal{S}, M/\mathcal{O}^{\times})$. Then the following (i) and (ii) are equivalent.*

(i) *Étale locally on S , we have either $fg^{-1} \in M/\mathcal{O}^{\times}$ or $f^{-1}g \in M/\mathcal{O}^{\times}$ in $M^{\text{gp}}/\mathcal{O}^{\times}$.*

(ii) *Locally on S for the weaker topology, we have either $fg^{-1} \in M/\mathcal{O}^{\times}$ or $f^{-1}g \in M/\mathcal{O}^{\times}$ in $M^{\text{gp}}/\mathcal{O}^{\times}$.*

Proof. Let $L_{\mathbb{Z}} = \mathbb{Z}^2$, let $\sigma_1 = \{(x, y) \in \mathbb{R}^2 \mid x \geq y \geq 0\}$, $\sigma_2 = \{(x, y) \in \mathbb{R}^2 \mid y \geq x \geq 0\}$, $\Sigma_i = \text{face}(\sigma_i)$ for $i = 1, 2$, $\Sigma = \Sigma_1 \cup \Sigma_2$. If the condition (i) is satisfied, then (f, g) belongs to $([\Sigma_1] \cup [\Sigma_2])(S)$ where \cup is the union of sheaves for the étale topology. By 2.5.10, this union $[\Sigma]$ is in fact the union of $[\Sigma_1]$ and $[\Sigma_2]$ as a sheaf for the weaker topology. Hence the condition (ii) is satisfied. \square

2.5.12. Assume we are one of the cases (a)–(d). Let Σ and Σ' be finite rational fans in $L_{\mathbb{R}}$ and assume that for each $\sigma' \in \Sigma'$, there exists $\sigma \in \Sigma$ such that $\sigma' \subset \sigma$. Then we have injective maps $\text{toric}(\Sigma')(S) \rightarrow \text{toric}(\Sigma)(S)$ and $[\Sigma'](S) \rightarrow [\Sigma](S)$. The morphism $\text{toric}_{\mathbb{Z}}(\Sigma') \rightarrow \text{toric}_{\mathbb{Z}}(\Sigma)$ of fs log schemes is log étale.

On the category of fs log schemes, the morphism $[\Sigma'] \rightarrow [\Sigma]$ is represented by log étale morphisms. In fact, if we are given an element $a \in [\Sigma](S)$, then Zariski locally on S , a comes from an element of $\text{toric}(\Sigma)(S)$ (2.5.9), that is, from a morphism $S \rightarrow \text{toric}_{\mathbb{Z}}(\Sigma)$ of fs log schemes, and $S \times_{[\Sigma]} [\Sigma'] = S \times_{\text{toric}_{\mathbb{Z}}(\Sigma)} \text{toric}_{\mathbb{Z}}(\Sigma')$ in this case.

In the case Σ' is a finite rational subdivision of Σ , the morphism $\text{toric}_{\mathbb{Z}}(\Sigma') \rightarrow \text{toric}_{\mathbb{Z}}(\Sigma)$ is the well known proper birational morphism in the classical toric geometry. In this case, on the category of fs log schemes, $[\Sigma'] \rightarrow [\Sigma]$ is represented by proper morphisms.

2.5.13. Assume we are one of the cases (a)–(d). Let Σ be a finite rational fan in $L_{\mathbb{R}}$. Define the topology of Σ by taking the sets $\text{face}(\sigma)$ for $\sigma \in \Sigma$ as a base of open sets.

An element a of $[\Sigma](S)$ induces a continuous map $\varphi : S \rightarrow \Sigma$ which sends $s \in S$ to the smallest cone σ of Σ such that the map $L_{\mathbb{Z}}^* \rightarrow M^{\text{gp}}/\mathcal{O}^{\times}$ induced by a sends $\sigma^{\vee} \subset L_{\mathbb{Z}}^*$ to $M/\mathcal{O}^{\times} \subset M^{\text{gp}}/\mathcal{O}^{\times}$ at s . This map is understood as follows. Locally on S , a lifts to an element \tilde{a} of $\text{toric}(\Sigma)(S)$. This \tilde{a} gives a continuous map $S \rightarrow \text{toric}_{\mathbb{Z}}(\Sigma)$. On the other hand, we have a continuous map $\text{toric}_{\mathbb{Z}}(\Sigma) \rightarrow \Sigma$ which sends $x \in \text{toric}_{\mathbb{Z}}(\Sigma)$ to the smallest element σ of Σ such that x belongs to $\text{toric}_{\mathbb{Z}}(\sigma)$. The map φ is locally the composition $S \rightarrow \text{toric}_{\mathbb{Z}}(\Sigma) \rightarrow \Sigma$ of continuous maps.

2.5.14. Consider the case (c) of 2.5.5. We call the log structure $M = \mathcal{O}_S^+ \cap \mathcal{O}_S^{\times}$ with the inclusion map $M \rightarrow \mathcal{O}_S^+$ on the site of open sets of S , the canonical log structure. This is a saturated log structure.

Note that for $s \in S$, the stalk $\mathcal{O}_{S,s}$ is a local ring and there is a valuation ring \mathcal{V} in the residue field $\kappa(s)$ of $\mathcal{O}_{S,s}$ such that the field of fractions of \mathcal{V} coincides with $\kappa(s)$ and such that the stalk $\mathcal{O}_{S,s}^+$ coincides with the inverse image of \mathcal{V} under $\mathcal{O}_{S,s} \rightarrow \kappa(s)$. Hence the stalk M_s is the inverse image of $\mathcal{V} \setminus \{0\}$ in $\mathcal{O}_{S,s}$. Hence $M^{\text{gp}} = \mathcal{O}_S^{\times}$. We have $M_s^{\text{gp}} = M_s \cup M_s^{-1}$.

Proposition 2.5.15. *For an adic space S endowed with the canonical log structure 2.5.14 and for a finite rational subdivision Σ' of Σ , the canonical map $[\Sigma'](S) \rightarrow [\Sigma](S)$ is bijective.*

2.5.16. We give a preparation for the proof of 2.5.15.

Let $\Sigma_{\text{val}} = \varprojlim \Sigma'$, where Σ' ranges over all finite rational subdivisions of Σ . Then Σ_{val} is identified with the set of all submonoids V of $L_{\mathbb{Z}}^*$ satisfying the following conditions (i) and (ii).

- (i) $V \cup (-V) = L_{\mathbb{Z}}^*$.
- (ii) $\sigma^{\vee} \subset V$ for some $\sigma \in \Sigma$.

In fact, for $x = (x_{\Sigma'})_{\Sigma'} \in \Sigma_{\text{val}}$, the corresponding V is given by $V = \cup_{\Sigma'} (x_{\Sigma'})^{\vee} \subset L_{\mathbb{Z}}^*$. Conversely, from V , we have the corresponding $x = (x_{\Sigma'})_{\Sigma'}$ where $x_{\Sigma'}$ the smallest element τ of Σ' such that $\tau^{\vee} \subset V$.

With this identification, the inverse limit topology on Σ_{val} is described as follows. The sets $\{V \in \Sigma_{\text{val}} \mid I \subset V\}$, where I ranges over all finite subsets of $L_{\mathbb{Z}}^*$, form a base of open sets.

2.5.17. We prove 2.5.15.

An element a of $[\Sigma](S)$ induces a continuous map $\varphi : S \rightarrow \Sigma$ (2.5.13). Furthermore, a induces a continuous map $\varphi_{\text{val}} : S \rightarrow \Sigma_{\text{val}}$ which sends $s \in S$ to the inverse image V of $(M/\mathcal{O}^\times)_s$ under $L_{\mathbb{Z}}^* \rightarrow (M^{\text{gp}}/\mathcal{O}^\times)_s$. (Note that since $M_s^{\text{gp}} = M_s \cup M_s^{-1}$, we have $L_{\mathbb{Z}}^* = V \cup (-V)$.) The composition $S \xrightarrow{\varphi_{\text{val}}} \Sigma_{\text{val}} \rightarrow \Sigma$ coincides with φ .

Let φ' be the composition $S \xrightarrow{\varphi_{\text{val}}} \Sigma_{\text{val}} \rightarrow \Sigma'$. For each $s \in S$, $a : L_{\mathbb{Z}}^* \rightarrow (M^{\text{gp}}/\mathcal{O}^\times)_s$ induces $\varphi'(s)^\vee \rightarrow (M/\mathcal{O}^\times)_s$. By the continuity of φ' , there is an open neighborhood U of s in S such that a induces $\varphi'(s)^\vee \rightarrow (M/\mathcal{O}^\times)_U$. Hence a comes from $[\Sigma'](S)$.

2.6 Level structures

We fix an element N of A which does not belong to the total constant field of F . Fix an integer $d \geq 1$.

2.6.1. Recall that for a scheme S over A and a Drinfeld module (\mathcal{L}, ϕ) over S of rank d , a *Drinfeld level N structure* on (\mathcal{L}, ϕ) is a homomorphism $\iota : (\frac{1}{N}A/A)^d \rightarrow \mathcal{L}$ which is compatible with the actions of A where A acts on \mathcal{L} via ϕ such that

$$\ker(\phi(N)) = \sum_{a \in (\frac{1}{N}A/A)^d} [\iota(a)]$$

as Cartier divisors. Here $[\iota(a)]$ is the image of the section $\iota(a) : S \rightarrow \mathcal{L}$ regarded as a Cartier divisor on \mathcal{L} . We will call Drinfeld level N structure simply a level N structure.

Lemma 2.6.2. *Let S be a normal scheme over A and let (\mathcal{L}, ϕ) be a Drinfeld module over S . Let U be a dense open subset of S , and let ι be a level N structure on $(\mathcal{L}, \phi)|_U$. Then ι extends uniquely to a level N structure of (\mathcal{L}, ϕ) .*

Proof. We may assume that $S = \text{Spec}(R)$ for a normal integral domain R with field of fractions K and the line bundle \mathcal{L} is trivialized. Then the coefficient of polynomial $\phi(N)(z)$ the in the highest degree $n = |N|^d$ (d is the rank of (\mathcal{L}, ϕ)) is a unit c of R . Since this polynomial is a product of polynomials over K of degree 1 and since R is normal, $\phi(N)(z) = c \prod_{i=1}^n (z - \alpha_i)$ for some $\alpha_i \in R$. Hence the A/NA -homomorphism $\iota : (\frac{1}{N}A/A)^d \rightarrow \mathcal{L}|_U$ is extended to $\iota : (\frac{1}{N}A/A)^d \rightarrow \mathcal{L}$ and we have $\ker(\phi(N)) = \text{div}(\phi(N)(z)) = \sum_i \text{div}(z - \alpha_i) = \sum_{a \in (\frac{1}{N}A/A)^d} [\iota(a)]$. \square

Lemma 2.6.3. *Let R be a normal local integral domain with maximal ideal m_R , residue field k , and field of fractions K . Consider a polynomial $f \in R[z]$ in one variable. Assume that the image of f in $k[z]$ is not zero and assume that f is a product of polynomials of degree 1 in $K[z]$. Then*

$$f = c \prod_{i=1}^m (z - a_i) \cdot \prod_{j=1}^n (1 - b_j z)$$

for some $m, n \geq 0$, $a_i \in R$ with $1 \leq i \leq m$, $b_j \in m_R$ with $1 \leq j \leq n$, and $c \in R^\times$.

Proof. By a simple limit argument, we are reduced to the case that R is a local ring of a finitely generated \mathbb{Z} -algebra. Assume we are in this case. We can write $f = g \prod_{i=1}^m (z - a_i)$ with $a_i \in R$ and with $g \in R[z]$ which has no root in R . The completion \hat{R} of R is also a normal integral domain as R is excellent. By the Weierstrass preparation theorem, $g = hu$ where h is a monic polynomial over \hat{R} and $u \in \hat{R}[[z]]^\times$. If h is of degree ≥ 1 , take a root α of h which is integral over \hat{R} . Then α is a root of g and hence belongs to K . By the normality of \hat{R} , it then belongs to $\hat{R} \cap K = R$. This is a contradiction. Hence $h = 1$ and $g = u$. Thus the constant term c of g is a unit of R . Write $g = c \cdot (1 + \sum_{i=1}^n c_i z^i)$ with $c_i \in R$. Then the inverse of any root β of g is a root of the monic polynomial $z^n + \sum_{i=1}^n c_i z^{n-i}$, and hence $\beta^{-1} \in R$. Since $\beta \notin R$, we have $\beta^{-1} \in m_R$. \square

Proposition 2.6.4. *Let (S, U) be a pair as in 1.2.7. If $((\mathcal{L}, \phi), \iota)$ is a generalized Drinfeld module over (S, U) of rank d with level N structure. Then $\iota : (\frac{1}{N}A/A) \rightarrow \mathcal{L}|_U$ extends uniquely to a map $(\frac{1}{N}A/A)^d \rightarrow \bar{\mathcal{L}}$, where $\bar{\mathcal{L}}$ is defined by the associated log structure of S (1.2.4).*

Proof. Working locally on S , we may assume $\mathcal{L} = \mathcal{O}_S$. Let $s \in S$. Note that the polynomial $\phi(N)$ modulo the maximal ideal of $\mathcal{O}_{S,s}$ is not zero. We apply Lemma 2.6.3 to $R = \mathcal{O}_{S,s}$ and $f = \phi(N)$. Then we see that for $a \in (\frac{1}{N}A/A)^d$, we have either $\iota(a) \in \mathcal{O}_{S,s}$ or $\iota(a)^{-1} \in \mathcal{O}_{S,s}$. We also have $\iota(a) \in j_*(\mathcal{O}_U)$. This proves that $\iota(a)$ belongs to $\bar{\mathcal{L}}$ at s . \square

To reduce various problems to the case of complete discrete valuation rings, the following lemma is useful.

Lemma 2.6.5. *Let R be an integral domain with field of fractions K and let I be an ideal of R . Assume R is an excellent ring. Let E be the set of all discrete valuation rings \mathcal{V} such that $R \subset \mathcal{V} \subset K$ and such that the maximal ideal $m_{\mathcal{V}}$ of \mathcal{V} contains I .*

(1) *Assume R is a normal. Assume $\{1 + x \mid x \in I\} \subset R^\times$ (note that I has this property if either R is a local ring and I is its maximal ideal, or if $R \xrightarrow{\cong} \varprojlim_n R/I^n$). Then $R = \bigcap_{\mathcal{V} \in E} \mathcal{V}$.*

(2) *Assume R/I is a non-zero reduced ring. Then $I = \bigcap_{\mathcal{V} \in E} m_{\mathcal{V}}$.*

Proof. We first prove the following (0).

(0) If R is a local ring with maximal ideal m , there is a discrete valuation ring \mathcal{V} such that $R \subset \mathcal{V} \subset K$ and $R \cap m_{\mathcal{V}} = m$.

In fact, let X be the normalization of the blowing-up of $\text{Spec}(R)$ along m and take a point x of X lying over m . Then m generates a principal ideal in $\mathcal{O}_{X,x}$ generated by an element t which is not a unit. Since $\mathcal{O}_{X,x}$ is normal and Noetherian, there is a prime ideal \mathfrak{p} of height one of $\mathcal{O}_{X,x}$ which contains t . Then the local ring \mathcal{V} of $\mathcal{O}_{X,x}$ at \mathfrak{p} has the desired property.

We prove (1). Let $f \in K$ and assume $f \notin R$. Let X be the blowing-up of $\text{Spec}(R)$ along the fractional ideal $R + Rf$. On X , for each point x of X , we have either $f \in \mathcal{O}_{X,x}$ or $f^{-1} \in \mathcal{O}_{X,x}$.

Since $g_*\mathcal{O}_X = R$, there is $x \in X$ such that $f \notin \mathcal{O}_{X,x}$. Since $X \rightarrow \text{Spec}(R)$ is proper, the image in $\text{Spec}(R)$ of the closure of x in X is closed and hence contains a maximal ideal m of R . By the assumption on I in (1), m contains I . Let $y \in X$ be the element of the closure of x in X whose image in $\text{Spec}(R)$ is m . Then $f \notin \mathcal{O}_{Y,y}$. Hence f^{-1} belongs to the maximal ideal m_y of $\mathcal{O}_{Y,y}$. By (0), there is a discrete valuation ring \mathcal{V} such that $\mathcal{O}_{Y,y} \subset \mathcal{V}$ and $\mathcal{O}_{Y,y} \cap m_{\mathcal{V}} = m_y$. We have $f^{-1} \in m_{\mathcal{V}}$ and hence $f \notin \mathcal{V}$.

(2) By the assumption, I is the intersection of all prime ideal \mathfrak{p} of R such that $I \subset \mathfrak{p}$. By (0) applied to the local ring of R at \mathfrak{p} , there is a discrete valuation ring \mathcal{V} such that $R \subset \mathcal{V} \subset K$ and $\mathfrak{p} = R \cap m_{\mathcal{V}}$. \square

In this paper, for a sheaf \mathcal{F} on the étale site, $\mathcal{F}_{\bar{s}}$ denotes the stalk of \mathcal{F} lying over the point s in the sense of étale topology.

Lemma 2.6.6. *Let (S, U) be as in 1.2.7 and let $(\mathcal{L}, \phi, \iota)$ be a generalized Drinfeld module of rank d with level N structure. Define the subsheaf E of $(\frac{1}{N}A/A)^d$ on the étale site of S to be the inverse image of \mathcal{L} under $\iota : (\frac{1}{N}A/A)^d \rightarrow \overline{\mathcal{L}}$ (2.6.4).*

(1) *Let $s \in S$. We have $E_s := \Gamma(\text{Spec}(\mathcal{O}_{S,s}), E) \xrightarrow{\cong} E_{\bar{s}}$. Let r be the rank of the fiber of (\mathcal{L}, ϕ) at s . Then the A/NA -module E_s is free of rank r . Let $E'_s = (\frac{1}{N}A/A)^d \setminus E_s$. If the line bundle \mathcal{L} is trivialized at s , the polynomial $\phi(N)(z)$ is expressed over $\mathcal{O}_{S,s}$ as*

$$\phi(N)(z) = c \prod_{a \in E_s} (z - \iota(a)) \cdot \prod_{a \in E'_s} (1 - \iota(a)^{-1}z)$$

for some unit c of $\mathcal{O}_{S,s}$, and $\iota(a)^{-1}$ belongs to the maximal ideal m_s of $\mathcal{O}_{S,s}$ for all $a \in E'_s$.

(2) *Let I be a quasi-coherent ideal of \mathcal{O}_S , and assume that \mathcal{O}_S/I is a sheaf of reduced rings, and that $(\mathcal{L}, \phi) \bmod I$ is a Drinfeld module of rank r over \mathcal{O}_S/I . Assume that the line bundle \mathcal{L} is trivialized. Then the image of the composite map*

$$(\frac{1}{N}A/A)^d \setminus E \xrightarrow{\iota} M_S^{-1}\mathcal{L}^\times = M_S^{-1} \xrightarrow{c} M_S \rightarrow \mathcal{O}_S,$$

where $c(f) = f^{-1}$, is contained in I .

Proof. (1) The bijectivity $E_s \xrightarrow{\cong} E_{\bar{s}}$ is clear. Trivialize the line bundle \mathcal{L} at s . By 2.6.3, we obtain the product formula for $\phi(N)(z)$ stated above and that $\sharp(E_s)$ has order $|N|^r$. To prove $E_s \cong (A/NA)^r$, we may assume that S is of finite type over A and hence is excellent. Take a discrete valuation ring \mathcal{V}' in the field of fractions of $\mathcal{O}_{S,s}$ such that $\mathcal{O}_{S,s} \subset \mathcal{V}'$ and $m_s \subset m_{\mathcal{V}'}$ (2.6.5). Let \mathcal{V} be the completion of \mathcal{V}' and let K be the field of fractions of \mathcal{V} . By replacing $\mathcal{O}_{S,s}$ by \mathcal{V} , we may assume that $\mathcal{O}_{S,s} = c\mathcal{V}$. Then we have an exact sequence

$$0 \rightarrow \psi[N] \xrightarrow{c} \phi[N] \rightarrow \Lambda/NA \rightarrow 0$$

of finite flat group schemes over K , where e exponential map of the lattice Λ . Concerning the level N -structure ι over K , for $a \in (\frac{1}{N}A/A)^d$ with $\iota(a) \in \psi[N]$, we have $a \in E_s$. Since the finite flat group scheme $\psi[N]$ has rank $|N|^r$ and this is equal to the order of E_s , we have that $\iota(a) \in \psi[N]$ if and only if $a \in E_s$. Hence the above exact sequence shows that $(\frac{1}{N}A/A)^d \cong \Lambda/N\Lambda$ as an A/NA -module. Hence $E_s \cong (A/NA)^r$ as an A/NA -module.

(2) We may assume that S is of finite type over A and hence is excellent. By 2.6.5 (2), we may assume $S = \text{Spec}(\mathcal{V})$ for a complete discrete valuation ring \mathcal{V} . Then this (2) follows from the formula for $\phi(N)(z)$ in (1) at the closed point s of $\text{Spec}(\mathcal{V})$, \square

A goal of the rest of this Section 2.6 is to prove Proposition 2.6.11 on automorphisms of $((\mathcal{L}, \phi), \iota)$.

Lemma 2.6.7. *Let S be a normal integral scheme over A with function field K , and let ϕ be a generalized Drinfeld module over S with line bundle \mathcal{O}_S . Let $N_1, N_2 \in A$ be such that N_1 is invertible on S and $N_2 \neq 0$. Suppose that $x \in K$ satisfies $\phi(N_1 N_2)x = 0$ and $\phi(N_2)x \neq 0$. Then $x^{-1} \in \Gamma(S, \mathcal{O}_S)$.*

Proof. First assume $N_2 = 1$. Since N_1 is invertible on S and

$$0 = \phi(N_1)(x) = N_1 x + \sum_{i=2}^n a_i x^i$$

for some $n \geq 2$ and $a_i \in \mathcal{O}_S$ for $2 \leq i \leq n$, we have that x^{-1} is integral over \mathcal{O}_S . Since S is normal, x^{-1} belongs to \mathcal{O}_S .

In general, assume that x^{-1} does not belong to \mathcal{O}_S . Since S is normal, there is a valuation ring $\mathcal{V} \subset K$ which dominates S such that x^{-1} does not belong to \mathcal{V} . Then x belongs to the maximal ideal of \mathcal{V} . Hence $\phi(N_2)x$ belongs to the maximal ideal of \mathcal{V} . Hence $(\phi(N_2)x)^{-1}$ does not belong to \mathcal{V} . But $\phi(N_1)\phi(N_2)x = 0$ and we have seen $(\phi(N_2)x)^{-1} \in \mathcal{O}_S$, which is a contradiction. \square

2.6.8. In the following 2.6.9–2.6.11, let (S, U) be as in 1.2.7 and let $((\mathcal{L}, \phi), \iota)$ be a generalized Drinfeld module over (S, U) of rank d with level N structure. Assume we are in one of the following situations (i) and (ii).

- (i) N has at least two prime divisors.
- (ii) N is invertible on S .

Proposition 2.6.9. *Assume we are in one of the following situations (i) and (ii).*

(i) *We are in the situation (i) of 2.6.8 and let $(\frac{1}{N}A/A)_*^d$ to be the non-empty subset of $(\frac{1}{N}A/A)^d$ consisting of all elements a such that the ideal $\{b \in A \mid ab = 0\}$ of A has at least two prime divisors.*

(ii) *We are in the situation (ii) of 2.6.8 and let $(\frac{1}{N}A/A)_*^d := (\frac{1}{N}A/A)^d \setminus \{0\}$.*

Then we have

$$\iota((\frac{1}{N}A/A)_{*,S}^d) \subset M_S^{-1} \mathcal{L}^\times \subset \overline{\mathcal{L}}.$$

Proof. We may assume that the line bundle \mathcal{L} is trivial. Let $a \in (\frac{1}{N}A/A)_*^d$. Working locally on S , let v be a maximal ideal of A which divides the ideal $\{b \in A \mid ab = 0\}$ and which is not in the image of $S \rightarrow \text{Spec}(A)$. Let $N_1 \in A$ generate the highest power of v dividing N , and let $N_2 = N/N_1$. Then the conditions of Lemma 2.6.7 are satisfied for $x = \iota(a)$, so $\iota(a)^{-1} \in \Gamma(S, \mathcal{O}_S)$, which implies the result. \square

Proposition 2.6.10. *Let the situation be as in 2.6.8. Then locally on S , there exists an $a \in (\frac{1}{N}A/A)^d$ such that $\iota(a)$ is a base of \mathcal{L} .*

Proof. Let $s \in S$, trivialize \mathcal{L} at s , and let E_s be as in 2.6.6. Since $E_s \cong (A/NA)^r$ as an A/NA -module with $r \geq 1$ by 2.6.6, there exists $a \in E_s$ which belongs to $(\frac{1}{N}A/A)_*^d$. For this a , since $a \in E_s$, $\iota(a)$ belongs to $\mathcal{O}_{S,s}$. On the other hand, $\iota(a)^{-1}$ belongs to $\mathcal{O}_{S,s}$ by 2.6.9. Hence $\iota(a) \in (\mathcal{O}_{S,s})^\times$. \square

Proposition 2.6.11. *Under the assumption in 2.6.8, the automorphism group of $((\mathcal{L}, \phi), \iota)$ is trivial.*

Proof. This follows from 2.6.10. \square

2.7 Log Drinfeld modules

We fix an element N of A which does not belong to the total constant field of F . Fix an integer $d \geq 1$. We prove basic results on log Drinfeld modules of rank d with N level structure.

2.7.1. Let S be a scheme with a log structure M . We say S is log regular if S is locally Noetherian, the log structure is fs, and the following (i) and (ii) are satisfied for every $s \in S$. Let I be the ideal of $\mathcal{O}_{S,\bar{s}}$ generated by the image of $M_{S,\bar{s}} \setminus \mathcal{O}_{S,\bar{s}}^\times \rightarrow \mathcal{O}_{S,\bar{s}}$.

- (i) $\mathcal{O}_{S,\bar{s}}/I$ is a regular local ring, and
- (ii) $\dim(\mathcal{O}_{S,\bar{s}}) = \dim(\mathcal{O}_{S,\bar{s}}/I) + \text{rank}_{\mathbb{Z}}((M_S^{\text{gp}}/\mathcal{O}_S^\times)_{\bar{s}})$.

For log regularity, see [20] for log structures on the Zariski site, and see [16, Exposé VI] and [8, Section 12.5] etc. for log structures on the étale site (which we use in this paper). We call a log regular log scheme a *log regular scheme* for brevity.

2.7.2. For a log regular scheme S , the following (1)–(4) hold by 4.1, 11.6, 8.2, and 3.1 of [20], in that order.

- (1) The underlying scheme of S is normal.
- (2) We have $M_S = \mathcal{O}_S \cap j_*(\mathcal{O}_U^\times) \subset j_*(\mathcal{O}_U)$.
- (3) Any log smooth scheme over S is log regular.

- (4) Let k be a perfect field, and suppose that S is a scheme over k of finite type with an fs log structure. Then S is log regular if and only if S is log smooth over k .

For a scheme S over A with a saturated log structure, we define the notion log Drinfeld module over S of rank d with level N structure as in 1.2.8.

2.7.3. We prove (1) of 1.2.9, that is, in the case the log structure of S is trivial, a log Drinfeld module over S of rank d with level N structure is equivalent to a Drinfeld module over S of rank d with level N structure.

Assume we are given a log Drinfeld module over S of rank d with level N structure. By definition, it comes from a log regular S' . The morphism $S \rightarrow S'$ factors as $S \rightarrow U \rightarrow S'$, where U is the part of S' with trivial log structure which is open in S' . On U , we have a Drinfeld module of rank d with level N structure in the usual sense. Since we pull it back, the original object is a Drinfeld module of rank d with level N structure. Conversely, assume that we have a Drinfeld module over S of rank d with level N structure. It is obtained by a morphism $S \rightarrow \mathcal{M}_N^d$ as the pullback of the universal object over the regular scheme \mathcal{M}_N^d , yielding a log Drinfeld module with level N structure.

2.7.4. Let (S, U) be as in 1.2.7 and let $((\mathcal{L}, \phi), \iota)$ be a generalized Drinfeld module over (S, U) of rank d with level N structure. We consider the following condition on divisibility.

(div) For every $a, b \in (\frac{1}{N}A/A)^d$, we have locally on S either $\text{pole}(a)\text{pole}(b)^{-1} \in M_S/\mathcal{O}_S^\times$ or $\text{pole}(b)\text{pole}(a)^{-1} \in M_S/\mathcal{O}_S^\times$ in $M_S^{\text{gp}}/\mathcal{O}_S^\times$.

Note that by 2.5.11, we have the same condition (div) if we replace “locally” (this means Zariski locally) by “étale locally”.

Theorem 2.7.5. *Let S be a log regular scheme over A , let U be the dense open set of S consisting of all points at which the log structure is trivial, and let $((\mathcal{L}, \phi), \iota)$ be a generalized Drinfeld module of rank d with level N structure. Then the condition (div) is satisfied.*

This 2.7.5 proves 1.2.10.

Since we can work étale locally on S by the remark at the end of 2.7.4, 2.7.5 follows from

Lemma 2.7.6. *Let S be a log regular scheme over a field, and assume it is strictly local. Let s be the closed point of S . Let $x, y \in M_{S,s}$, and assume that $x^{-1} + y^{-1} \in \mathcal{O}_{S,s} \cup M_{S,s}^{-1}$. Then either $xy^{-1} \in M_{S,s}$ or $yx^{-1} \in M_{S,s}$.*

Proof. Let m_s be the maximal ideal of $\mathcal{O}_{S,s}$. We may assume that $x, y \in m_s$. If

$$x^{-1} + y^{-1} = c \in \mathcal{O}_{S,s},$$

then $y = x(cx - 1)^{-1}$ with $cx - 1$ a unit. Hence we may assume that $x^{-1} + y^{-1} = z^{-1}$ for some $z \in M_{S,s}$.

The completion R of $\mathcal{O}_{S,s}$ is isomorphic to $k[[P]][[T_1, \dots, T_n]]$ for some field k and sharp fs monoid P such that the log structure of $\text{Spec}(R)$ is given by $P \rightarrow R$ [20, 3.1]. It is sufficient to prove that if $\gamma_1, \gamma_2, \gamma_3 \in P$ and $u_1, u_2, u_3 \in R^\times$ are such that

$$u_1\gamma_1^{-1} + u_2\gamma_2^{-1} = u_3\gamma_3^{-1},$$

then either $\gamma_1\gamma_2^{-1} \in P$ or $\gamma_2\gamma_1^{-1} \in P$. Assume this is not satisfied.

As is well known, if $\gamma \in P^{\text{gp}}$ is such that $f(\gamma) \geq 0$ for every homomorphism $f: P \rightarrow \mathbb{N}$, then $\gamma \in P$. Taking $\gamma = \gamma_1\gamma_2^{-1}$ and also $\gamma = \gamma_2\gamma_1^{-1}$, we see that there exist homomorphisms $g, h: P \rightarrow \mathbb{N}$ such that $g(\gamma_1) > g(\gamma_2)$ and $h(\gamma_1) < h(\gamma_2)$. The pair (g, h) induces $P \rightarrow \mathbb{N} \times \mathbb{N}$ and hence induces a ring homomorphism

$$k[[P]][[T_1, \dots, T_n]] \rightarrow k[[\mathbb{N} \times \mathbb{N}]] = k[[t_1, t_2]],$$

where each T_i is sent to 0. Let $a = g(\gamma_1)$, $b = g(\gamma_2)$, $c = h(\gamma_1)$, $d = h(\gamma_2)$ so that $a, b, c, d \in \mathbb{N}$ with $a > b$ and $c < d$. Write the image of γ_3 in $k[[t_1, t_2]]$ as $t_1^e t_2^f$ with $e, f \in \mathbb{N}$. Then there are $v_1, v_2, v_3 \in k[[t_1, t_2]]^\times$ such that

$$v_1 t_1^{-a} t_2^{-c} + v_2 t_1^{-b} t_2^{-d} = v_3 t_1^{-e} t_2^{-f}.$$

Thus we have

$$v_1 t_2^{d-c} + v_2 t_1^{a-b} = v_3 t_1^{a-e} t_2^{d-f}$$

with $a - b > 0$ and $d - c > 0$, which is impossible. \square

Remark 2.7.7. Without the log regularity, the condition (div) in 2.7.4 need not be satisfied. See 5.4.10.

Proposition 2.7.8. *Let $((\mathcal{L}, \phi), \iota)$ be a log Drinfeld module of rank d with level N structure over a scheme S over A with a saturated log structure. Suppose that either N has at least two prime divisors or N is invertible on S . Then locally on S , there exists an $a \in (\frac{1}{N}A/A)^d$ such that $\iota(a) \in \mathcal{L}^\times$.*

Proof. This is reduced the case S is log regular and hence to 2.6.10. \square

Proposition 2.7.9. *Under the assumptions of Proposition 2.7.8, the automorphism group of $((\mathcal{L}, \phi), \iota)$ is trivial.*

Proof. This follows from 2.7.8. \square

2.8 Bruhat-Tits buildings

Like the toroidal compactifications considered in [3] and [6], our toroidal compactifications are associated to cone decompositions. Our cone decompositions base on the decomposition of the Bruhat-Tits building of $PGL_n(F_\infty)$ into simplices.

In this Section 3.1, we review the Bruhat-Tits buildings of $PGL_n(E)$ in a style which matches this paper. At the end of this Section 2.8, we give a proposition 2.8.21 via which Bruhat-Tits buildings are related to Drinfeld modules.

Let E be a complete discrete valuation field with finite residue field \mathbb{F}_q . Let O_E be the valuation ring of E and let m_E be the maximal ideal of O_E . Let $|\cdot| : E \rightarrow \mathbb{R}_{\geq 0}$ be the standard absolute value. Let $n \geq 1$.

2.8.1. Let V be an E -vector space of finite dimension n . By a *norm* on V , we mean a map $\mu : V \rightarrow \mathbb{R}_{\geq 0}$ which satisfies the following two equivalent conditions.

- (i) There exists a basis $(e_i)_{1 \leq i \leq n}$ of V such that there are $(s_i)_{1 \leq i \leq n} \in \mathbb{R}_{> 0}^d$ for which

$$\mu \left(\sum_{i=1}^n x_i e_i \right) = \max\{s_i |x_i| \mid 1 \leq i \leq n\}$$

for all $x_i \in E$.

- (ii) We have

- (a) $\mu(ax) = |a|\mu(x)$ for $a \in E$ and $x \in V$,
- (b) $\mu(x + y) \leq \max(\mu(x), \mu(y))$ for $x, y \in V$, and
- (c) $\mu(x) > 0$ for all $x \in E \setminus \{0\}$.

See [9] for the equivalence. Note that in (b), we have $\mu(x + y) = \max(\mu(x), \mu(y))$ if $\mu(x) \neq \mu(y)$. We will call a basis of V satisfying (i) an *orthonormal basis* for μ .

2.8.2. Let $|BT_n|$ be the set of all homothety classes of norms (2.5.1) on the E -vector space E^n .

The topology of $|BT_n|$ is defined by the embedding

$$|BT_n| \xrightarrow{\hookrightarrow} \prod_{(x,y)} \mathbb{R}_{> 0}; \mu \mapsto \mu(x)\mu(y)^{-1},$$

where (x, y) ranges over all elements of $(E^n - \{0\}) \times (E^n - \{0\})$ and for $\mu \in |BT_n|$, $\mu(x)/\mu(y)$ means $\tilde{\mu}(x)/\tilde{\mu}(y)$ for a norm $\tilde{\mu}$ with class μ .

We have a natural action of $PGL_n(E)$ on $|BT_n|$: The class $g \in PGL_n(E)$ of $\tilde{g} \in GL_n(E)$ sends the class $\mu \in |BT_n|$ of a norm $\tilde{\mu}$ to the class $g\mu \in |BT_n|$ of the norm $x \mapsto \tilde{\mu}(\tilde{g}^{-1}(x))$.

2.8.3. The Bruhat-Tits building $|BT_n|$ is identified with the geometric realization of a simplicial complex BT_n which is also called the Bruhat-Tits building.

The set of 0-simplices of BT_n is introduced in 2.8.4. The set of general simplices of BT_n is introduced in 2.8.7. For each simplex S of BT_n , its geometric realization $|BT_n|(S) \subset |BT_n|$ is introduced in 2.8.10.

2.8.4. A 0-simplex of BT_n is the class of an O_E -lattice in E^n . Here by an O_E -lattice in E^n , we mean a finitely generated O_E -submodule of E^n which generates E^n over E . Two O_E -lattices L and L' are said equivalent if $L' = aL$ for some $a \in E^\times$.

For an O_E -lattice L in E^n , we have the associated norm μ_L on E^n defined as

$$\mu_L(x) = \min\{|a| \mid a \in E, x \in aL\}.$$

By $\text{class}(L) \mapsto \text{class}(\mu_L)$, we regard a 0-simplex of BT_n as an element of $|BT_n|$.

2.8.5. We consider a condition $C(S)$ for a subset S of $|BT_n|$ and a condition $C(S, \mu)$ for a subset S of $|BT_n|$ and for $\mu \in |BT_n|$.

$C(S)$: For each pair (x, y) of non-zero elements of E^n , we have either $\mu(x) \geq \mu(y)$ for all $\mu \in S$ or $\mu(x) \leq \mu(y)$ for all $\mu \in S$. Here $\mu(x) \geq \mu(y)$ (resp. $\mu(x) \leq \mu(y)$) means $\tilde{\mu}(x) \geq \tilde{\mu}(y)$ (resp. $\tilde{\mu}(x) \leq \tilde{\mu}(y)$) for a norm $\tilde{\mu}$ with class μ .

$C(S, \mu)$: If x and y are non-zero elements of E^n and if $\mu'(x) \geq \mu'(y)$ for all $\mu' \in S$, then we have $\mu(x) \geq \mu(y)$.

Proposition 2.8.6. *For a non-empty set S of 0-simplices of BT_n , the following conditions (i) – (iii) are equivalent.*

(i) $C(S)$.

(ii) For O_E -lattices L and L' in E^n whose classes belong to S , we have either $L \subset L'$ or $L' \subset L$.

(iii) There are O_E -lattices L^0, \dots, L^r in E^n such that S is the set of equivalence classes of L^0, \dots, L^r and such that $L^0 \supseteq L^1 \supseteq \dots \supseteq L^r \supseteq m_E L^0$.

Proof. (i) \Rightarrow (ii). Assume (ii) is not satisfied. Then there are O_E -lattices L, L' in E^n whose classes belong to S such that $L \not\subset L'$ and $L' \not\subset L$. Take $x, y \in E^n$ such that $x \in L, x \notin L', y \in L', y \notin L$. Then $\mu_L(x) < \mu_L(y)$ and $\mu_{L'}(x) > \mu_{L'}(y)$.

(ii) \Rightarrow (iii). Fix an O_E -lattice L^0 in E^n whose class belongs to S . For each element s of S , take the O_E -lattice $L(s)$ in E^n with class s such that $L(s) \subset L^0$ and $L(s) \not\subset m_E L^0$. Then the set $\{L(s) \mid s \in S\}$ is totally ordered for the inclusion and $L(s) \supset m_E L^0$. This proves that (iii) is satisfied.

(iii) \Rightarrow (i). Take O_E -lattices L^0, \dots, L^r satisfying the condition (iii). Let $L^{i+(r+1)t} = m_E^t L^i$ for $0 \leq i \leq r$ and $t \in \mathbb{Z}$. Then L^i for $i \in \mathbb{Z}$ form a decreasing filtration on E^n and $\cup_{i \in \mathbb{Z}} L^i = E^n$. Let x, y be non-zero elements of E^n . Let $i, j \in \mathbb{Z}$ be such that $x \in L^i, x \notin L^{i+1}, y \in L^j, y \notin L^{j+1}$. If $i \leq j$ (resp. $i \geq j$), we have $\mu_{L^m}(x) \geq \mu_{L^m}(y)$ (resp. $\mu_{L^m}(x) \leq \mu_{L^m}(y)$) for all $m \in \mathbb{Z}$. \square

2.8.7. A simplex of BT_n is a non-empty set of 0-simplices of BT_n satisfying the equivalent conditions in 2.8.6. By the condition (iii) there, we see that a simplex is a finite set (it is an r -simplex, where r is as in (iii) there).

The group $PGL_n(E)$ acts on BT_n in the natural way and this action is compatible with the action of $PGL_n(E)$ in $|BT_n|$. For each r , the action of $PGL_n(E)$ on the set of all r -simplices in BT_n is transitive.

Lemma 2.8.8. *Let $(\mu_i)_{i \in I}$ be a non-empty finite family of norms on E^n such that the set S of the classes of μ_i satisfies $C(S)$. Let $a_i \in \mathbb{R}_{>0}$ ($i \in I$). Then $\mu := \sum_i a_i \mu_i$ is a norm on E^n .*

Proof. Let x and y be non-zero elements of E^n . We prove that $\mu(x+y) \leq \max(\mu(x), \mu(y))$. We may assume that $\mu_i(x) \geq \mu_i(y)$ for all i . We have $\mu(x+y) = \sum_i a_i \mu_i(x+y) \leq \sum_i a_i \max(\mu_i(x), \mu_i(y)) = \sum_i a_i \mu_i(x) = \mu(x)$. \square

Proposition 2.8.9. *Let S be a simplex of BT_n and let $\mu \in |BT_n|$. Then the following two conditions are equivalent.*

(i) $C(S, \mu)$.

(ii) $\tilde{\mu} = \sum_{s \in S} a_s \tilde{s}$ for some $a_s \in \mathbb{R}_{\geq 0}$, where $\tilde{\mu}$ is a norm with class μ and \tilde{s} is a norm with class s .

Proof. (i) \Rightarrow (ii). Let L^i ($i \in \mathbb{Z}$) be as in the proof of the part (iii) \Rightarrow (i) of 2.8.6. For $m \in \mathbb{Z}$, let Y_m be the set of all $x \in E^n$ such that $x \in L^m$ and $x \notin L^{m+1}$. Let P be the set of all maps $E^n \rightarrow \mathbb{R}$ such that $f(0) = 0$, $f(x) = f(y)$ if $m \in \mathbb{Z}$ and $x, y \in Y_m$, and $f(ax) = |a|f(x)$ if $a \in E$ and $x \in E^n$. For $m \in \mathbb{Z}$, take an element x_m of Y_m . Then we have an isomorphism $P \xrightarrow{\cong} \mathbb{R}^{r+1}$; $f \mapsto (f(x_m))_{0 \leq m \leq r}$. On the other hand, we have an isomorphism $\mathbb{R}^{r+1} \xrightarrow{\cong} P$; $(a_i)_{0 \leq i \leq r} \mapsto \sum_{i=0}^r a_i \mu_{L^i}$. Furthermore, a simple computation shows that if f is the image of $(a_i)_{0 \leq i \leq r} \in \mathbb{R}^{r+1}$ in P , then $a_i = (f(x_{i-1}) - f(x_i))(q-1)^{-1}$. Hence if $f(x_i) \geq f(x_{i+1})$ for all $i \in \mathbb{Z}$, then $a_i \geq 0$ for all $0 \leq i \leq r$ and $a_i > 0$ for some i .

Claim. $\tilde{\mu} \in P$. Furthermore, $\tilde{\mu}(x_i) \geq \tilde{\mu}(x_{i+1})$ for all i .

We prove Claim. If $x, y \in Y_m$, $\mu_{L^i}(x) = \mu_{L^i}(y)$ for all i , and hence the condition $C(S, \mu)$ shows that $\tilde{\mu}(x) = \tilde{\mu}(y)$. Furthermore, since $\mu_{L^i}(x_m) \geq \mu_{L^i}(x_{m+1})$ for all i , the condition $C(S, \mu)$ shows that $\tilde{\mu}(x_m) \geq \tilde{\mu}(x_{m+1})$. This proves Claim.

Hence $\tilde{\mu} = \sum_{i=0}^r a_i \mu_{L^i}$ for some $a_i \geq 0$ such that $a_i > 0$ for some i .

The implication (ii) \Rightarrow (i) is clear. \square

2.8.10. For a simplex S of BT_n , the geometric realization $|BT_n|(S) \subset |BT_n|$ of S is the set of all $\mu \in |BT_n|$ satisfying the equivalent conditions in 2.8.9.

If s is a 0-simplex, then $|BT_n|(\{s\})$ is the one point set $\{s\}$.

2.8.11. We have a canonical map

$$\mathbb{R}_{>0}^n \rightarrow |BT_n|; s \mapsto \text{class}(\mu_s); \quad \mu_s(x) = \max_i (|x_i|s_i) \quad \text{for } x \in E^n.$$

It induces an injection $\mathbb{R}_{>0}^n/\mathbb{R}_{>0} \xrightarrow{\hookrightarrow} |BT_n|$ where the multiplicative group $\mathbb{R}_{>0}$ acts on $\mathbb{R}_{>0}^n$ diagonally. Let $|AP_n| \subset |BT_n|$ be the image of this injection.

The map $PGL_n(E) \times \mathbb{R}_{>0}^n \rightarrow |BT_n|$; $(g, s) \mapsto g\text{class}(\mu_s)$ is surjective as is seen from the condition (i) in 2.8.1 on a norm. The topology of $|BT_n|$ is also described as the quotient topology of the topology of $PGL_n(E) \times \mathbb{R}_{>0}^n$ under this surjection.

For a norm μ on E^n whose class belongs to $|AP_n|$, $\mu = \mu_s$ for $s = (\mu(e_i))_{1 \leq i \leq n}$ where $(e_i)_i$ is the standard base of E^n . Hence if $(\mu^{(i)})_i$ is a finite family of norms on E^n such that the class of $\mu^{(i)}$ belongs to $|AP_n|$ for every i and such that the set S of the classes of $\mu^{(i)}$ satisfies $C(S)$, then the norm $\mu = \sum_i a_i \mu^{(i)}$ for $a_i \in \mathbb{R}_{>0}$ (2.8.8) is described as $\mu = \mu_s$, where $s = \sum_i a_i s^{(i)} \in \mathbb{R}_{>0}^n$ with $s^{(i)}$ the element of $\mathbb{R}_{>0}^n$ such that $\mu^{(i)} = \mu_{s^{(i)}}$.

Remark 2.8.12. For a finite family of elements $s^{(i)} = (s_j^{(i)})_{1 \leq j \leq n}$ of $\mathbb{R}_{>0}^n$ and for $s = \sum_i s^{(i)} \in \mathbb{R}_{>0}^n$, we need not have $\mu_s = \sum_i \mu_{s^{(i)}}$ unless the condition $C(S)$ for $S = \{\text{class}(\mu_{s^{(i)}})\}$ is satisfied. For example, if $n = 2$ and $s^{(1)} = (q, 1)$, $s^{(2)} = (1, q)$, then for $s = s^{(1)} + s^{(2)} = (q + 1, q + 1)$, $\mu_s(1, 1) = q + 1$ does not coincide with $\mu_{s^{(1)}}(1, 1) + \mu_{s^{(2)}}(1, 1) = 2q$.

2.8.13. For an O_E -lattice L in E^n , the class of L in BT_n belongs to $|AP_n|$ if and only if $L = \bigoplus_{i=1}^n m_E^{a(i)} e_i$ for some $a(i) \in \mathbb{Z}$ where $(e_i)_i$ denotes the standard base of E^n . For this L , $\mu_L = \mu_s$ where $s = s(L) := (q^{a(1)}, \dots, q^{a(n)}) \in \mathbb{R}_{>0}^n$.

Let AP_n be the simplicial complex whose simplex is a simplex S of BT_n such that $S \subset |AP_n|$ in $|BT_n|$. For a simplex S of AP_n , let $\sigma(S)$ be the subcone of $\mathbb{R}_{\geq 0}^n$ generated by $s(L) \in \mathbb{R}_{>0}^n$ for O_E -lattices L in E^n whose classes belong to S .

The subset $|BT_n|(S)$ for a simplex S in BT_n is characterized by the following (i) and (ii). (Note that every simplex of BT_n is written as gS for some $g \in PGL_n(E)$ and for some simplex S of AP_n .)

(i) For $g \in PGL_n(E)$, we have $|BT_n|(gS) = g(|BT_n|(S))$.

(ii) Let S be a simplex of AP_n and let $\sigma(S)$ be the associated subcone of $\mathbb{R}_{\geq 0}^n$. Then we have a bijection $\sigma(S) \setminus \{(0, \dots, 0)\} \rightarrow |BT_n|(S)$; $s \mapsto \text{class}(\mu_s)$.

In the theory of Bruhat-Tits buildings, $|BT_n|(S)$ for simplices S of AP_n are called apartments in $|AP_n|$.

When S ranges over all elements of AP_n , the above cones $\sigma(S)$ form a cone decomposition of $\mathbb{R}_{>0}^n \cup \{(0, \dots, 0)\}$. This cone decomposition will be the base of the cone decompositions for our toroidal compactifications.

Proposition 2.8.14. Let $I = \{(h, i, j) \in \mathbb{Z}^3 \mid 1 \leq j < i \leq n\}$. For a map $\alpha : I \rightarrow \{\mathbb{R}_{\leq 0}, \{0\}, \mathbb{R}_{\geq 0}\}$, let

$$c(\alpha) := \{s \in \mathbb{R}_{\geq 0}^n \mid q^h s_j - s_i \in \alpha(h, i, j) \text{ for all } (h, i, j) \in I\}.$$

Let Σ be the set of cones $c(\alpha)$ for these maps α . Then we have a bijection $S \mapsto \sigma(S)$ from AP_n to Σ .

This is easy to see.

Example 2.8.15. By the standard $(n - 1)$ -simplex of BT_n , we mean the $(n - 1)$ -simplex S consisting of the classes of the O_E -lattices L_i ($0 \leq i \leq n - 1$) defined by

$$L^i := \sum_{j=1}^{n-i} O_E e_j + \sum_{j=n-i+1}^n m_E e_j.$$

We have

$$\sum_{i=1}^d O_E e_i = L^0 \supset L^1 \supset \dots \supset L^{n-1} \supset m_E L^0.$$

We have

$$\sigma(S) = \{s \in \mathbb{R}_{\geq 0}^n \mid s_1 \leq s_2 \leq \dots \leq s_n \leq q s_1\}$$

and this is the set of all $s \in \mathbb{R}_{\geq 0}^n$ such that for $h \in \mathbb{Z}$ and for $1 \leq i < j \leq n$, $q^h s_i - s_j \geq 0$ if $h \geq 1$ and $q^h s_i - s_j \leq 0$ if $h \leq 0$.

Example 2.8.16. In the case $n = 2$, the cone decomposition of $\mathbb{R}_{\geq 0}^2$ given by $\sigma(S)$ for simplices S of AP_2 consists of the cones $\sigma(S_h) = \{s \in \mathbb{R}_{\geq 0}^2 \mid q^{h-1} s_1 \leq s_2 \leq q^h s_1\}$ ($h \in \mathbb{Z}$) where $S_h = \{\text{class}(O_E e_1 + m_E^{h-1} e_2), \text{class}(O_E e_1 + m_E^h e_2)\}$, and the cones $\sigma(\{\text{class}(L_h)\}) = \{s \in \mathbb{R}_{\geq 0}^2 \mid s_2 = q^h s_1\}$ where $L_h = O_E e_1 + \pi^h O_E e_2$.

2.8.17. We prove that $|BT_n| = \cup_S |BT_n|(S)$, where S ranges over all simplices of BT_n .

Since the map $PGL_n(E) \times |AP_n| \rightarrow |BT_n|$; $(g, \mu) \mapsto g\mu$ is surjective, it is sufficient to prove that each $\mu \in |AP_n|$ belongs to $\cup_S |BT_n|(S)$. Take a simplex S of AP_n such that a norm with class μ belongs to $\sigma(S)$. Then $\mu \in |BT_n|(S)$.

Lemma 2.8.18. Let S be a simplex of BT_n , let $\tilde{\mu} = \sum_{s \in S} a_s \tilde{s}$, where $a_s > 0$ and \tilde{s} is a norm with class s , and let $\mu \in |BT_n|$ be the class of $\tilde{\mu}$.

- (1) For $\mu' \in |BT_n|$, $\mu' \in |BT_n|(S)$ if and only if $C(\{\mu\}, \mu')$ is satisfied.
- (2) The set S coincides with the set of all 0-simplices s of BT_n such that $C(\{\mu\}, s)$ is satisfied.

Proof. (1) Let L^i and Y_i for $i \in \mathbb{Z}$ be as in the proof of 2.8.9. If $x \in Y_i$ and $y \in Y_j$, then $\mu(x) \geq \mu(y)$ if and only if $i \leq j$. For $\mu' \in |BT_n|$, the condition $\mu' \in |BT_n|(S)$ is equivalent to the condition that if $x \in Y_i$ and $y \in Y_j$ and $i \leq j$, then $\mu'(x) \geq \mu'(y)$. This proves (1).

(2) Let s be a 0-simplex of BT_n . If $s \in S$, $C(\{\mu\}, s)$ is satisfied by (1). Conversely, assume $C(\{\mu\}, s)$ is satisfied. Then $s \in |BT|(S)$. This implies $s \in S$. \square

Proposition 2.8.19. Let S be a subset of $|BT_n|$. Then the condition $C(S)$ is satisfied if and only if $S \subset |BT_n|(S')$ for some simplex S' in BT_n .

Proof. The if part is clear. We prove the only if part. Assume $C(S)$ is satisfied. For each $s \in S$, let $S'(s)$ be the set of all 0-simplices s' of BT_n such that $C(\{s\}, s')$ is satisfied. Then $C(S \cup \cup_{s \in S} S'(s))$ is satisfied. Hence $C(\cup_{s \in S} S'(s))$ is satisfied and hence $S' := \cup_{s \in S} S'(s)$ is a simplex. We have $S \subset |BT_n|(S')$ by 2.8.18. \square

2.8.20. We prove that $|BT_n|(S) \cap |BT_n|(S') = |BT_n|(S \cap S')$ for simplices S, S' of BT_n such that $S \cap S' \neq \emptyset$.

The inclusion $|BT_n|(S) \cap |BT_n|(S') \supset |BT_n|(S \cap S')$ is clear. We prove the other inclusion. Let $\mu \in |BT_n|(S) \cap |BT_n|(S')$. Let S'' be the set of all 0-simplices s of BT_n such that $C(\{\mu\}, s)$ is satisfied. Then $S'' \subset S \cap S'$ and $\mu \in |BT_n|(S'')$ by 2.8.18.

We take $E = F_\infty$.

Proposition 2.8.21. *Let \mathcal{V} be a complete valuation ring of height one over A as in 2.3.3. Let ψ be a Drinfeld module over \mathcal{V} of rank r with trivial line bundle, and let Λ be a $\psi(A)$ -lattice (2.3.3) in K^{sep} . Then there exists a unique norm μ on the F_∞ -vector space $F_\infty \otimes_A \Lambda$, where A acts on Λ via ψ , having the property that if $z \in \bar{K}$ and $a \in A \setminus \{0\}$ are such that $\psi(a)z$ is a nonzero element of Λ , then*

$$-v_{\bar{K}}(z) = |a|^{-r} \mu(\psi(a)z)^r.$$

Proof. Let $a \in A$ and $z \in \bar{K}$, and write

$$\psi(a)(z) = \sum_{i=1}^m a_i z^i \quad \text{with } a_1 = a, m = |a|^r, a_m \in \mathcal{V}^\times.$$

If $v_{\bar{K}}(z) < 0$, then since $a_i \in \mathcal{V}$ for each i and $a_m \in \mathcal{V}^\times$, we have $\psi(a)(z) = uz^{|a|^r}$ for some $u \in \mathcal{V}^\times$, so

$$v_{\bar{K}}(\psi(a)z) = |a|^r v_{\bar{K}}(z).$$

Since $v_{\bar{K}}$ takes negative values on nonzero elements of Λ by Lemma 2.3.4, there is then a well-defined map $\mu: \Lambda \rightarrow \mathbb{R}_{\geq 0}$ given by $\mu(\lambda) = (-v_{\bar{K}}(\lambda))^{1/r}$ and satisfying $\mu(\psi(a)\lambda) = |a|\mu(\lambda)$ for all $a \in A$ and $\lambda \in \Lambda$. This map μ uniquely extends to a map $\mu: F \otimes_A \Lambda \rightarrow \mathbb{R}_{\geq 0}$ such that $\mu(a \otimes \lambda) = |a|\mu(\lambda)$ for all $a \in F$ and $\lambda \in \Lambda$. By continuity, μ extends uniquely to a continuous map $\mu: F_\infty \otimes_A \Lambda \rightarrow \mathbb{R}_{\geq 0}$. This μ satisfies the condition (ii) of a norm in 2.8.1.

If $z \in \bar{K}$ and $a \in A$ are such that $\psi(a)z \in \Lambda \setminus \{0\}$, then $v_{\bar{K}}(\psi(a)z) < 0$ by 2.3.4, which forces $v_{\bar{K}}(z) < 0$. We then have

$$|a|^r v_{\bar{K}}(z) = v_{\bar{K}}(\psi(a)z) = -\mu(\psi(a)z)^r.$$

The uniqueness of the norm μ is evident. \square

3 Cone decompositions

In this section, we suppose that $A = \mathbb{F}_q[T]$.

3.1 Cone decompositions and torsion points. 1

In this Section 3.1, after we consider Bruhat-Tits buildings more, we state the results 3.1.6, 3.1.8, 3.1.9, 3.1.10 concerning torsion points of generalized Drinfeld modules over complete valuation rings of height one. The proofs of these results are given in Sections 3.3 later.

The following proposition will be applied to the study of generalized Drinfeld modules in the case $A = \mathbb{F}_q[T]$, through 2.3.5 and 2.8.21.

Proposition 3.1.1. *Assume $A = \mathbb{F}_q[T]$. Let Λ be a free A -module of finite rank n , and let μ be a norm on $F_\infty \otimes_A \Lambda$.*

- (1) *For a family $(\lambda_i)_{1 \leq i \leq n}$ of elements of Λ , the following two properties are equivalent.*
 - (i) *The collection $(\lambda_i)_i$ is an A -basis of Λ such that $\mu(\lambda_i) \leq \mu(\lambda_{i+1})$ for $1 \leq i \leq n-1$ and is orthonormal as an F_∞ -basis of $F_\infty \otimes_A \Lambda$ in the sense of (2.8.1).*
 - (ii) *For each i with $1 \leq i \leq n$, we have $\lambda_i \notin \Lambda_{i-1} := \sum_{j=1}^{i-1} A\lambda_j$, and λ_i has the smallest norm among $\Lambda \setminus \Lambda_{i-1}$.*
- (2) *There is a family $(\lambda_i)_{1 \leq i \leq n}$ satisfying the equivalent conditions in (1).*
- (3) *Let $(\lambda_i)_{1 \leq i \leq n}$ be a family of elements of Λ satisfying the equivalent conditions in (1). Let $\lambda'_i = \sum_{j=1}^n a_{ij}\lambda_j$ ($1 \leq i \leq n$) with $a_{ij} \in A$. Then $(\lambda'_i)_{1 \leq i \leq n}$ satisfies the equivalent conditions in (1) if and only if the following conditions (i)–(iii) are satisfied.*
 - (i) $a_{ij} = 0$ if $\mu(\lambda_j) < \mu(\lambda_i)$.
 - (ii) $\mu(a_{ij}\lambda_j) \leq \mu(\lambda_i)$ if $\mu(\lambda_j) \leq \mu(\lambda_i)$.
 - (iii) Let $0 \leq k < k+m \leq n$ and assume that $\mu(\lambda_i) = \mu(\lambda_{k+1})$ if and only if $k+1 \leq i \leq k+m$. Then $(a_{s+i, s+j})_{1 \leq i, j \leq m} \in GL_m(\mathbb{F}_q)$.
- (4) *If $(\lambda_i)_{1 \leq i \leq n}$ and $(\lambda'_i)_{1 \leq i \leq n}$ are families of elements of Λ satisfying the equivalent conditions in (1), we have $\mu(\lambda_i) = \mu(\lambda'_i)$ for $1 \leq i \leq n$.*

Proof. It is clear that for part (2) we can find $(\lambda_i)_i$ satisfying the condition (ii) of part (1) recursively. The implication (i) \Rightarrow (ii) of part (1) is easily seen. We prove that (ii) \Rightarrow (i). Let $(\lambda_i)_i$ be as in (ii). Let

$$\Lambda' = \left(\sum_{i=1}^{n-1} F\lambda_i \right) \cap \Lambda.$$

By induction on n , the tuple $(\lambda_i)_{1 \leq i \leq n-1}$ is an A -basis of Λ' and is an orthonormal basis (2.8.1) for the restriction of μ to $F_\infty \otimes_A \Lambda'$.

Claim 1. $(\lambda_i)_{1 \leq i \leq n}$ is an orthonormal basis (2.8.1) for μ .

We prove Claim 1. Since λ_i has the smallest norm among elements of $\Lambda \setminus \Lambda_i$ by (ii), we have in particular that $\mu(\lambda_i) \leq \mu(\lambda_{i+1})$ for $i \leq n - 1$. It remains to prove that $(\lambda_i)_i$ is an orthonormal basis for μ .

Given $(x_i)_{1 \leq i \leq n} \in F_\infty^n$, we must show that

$$\mu \left(\sum_{i=1}^n x_i \lambda_i \right) = \max\{|x_i| \mu(\lambda_i) \mid 1 \leq i \leq n\}.$$

By induction, we have this if $x_n = 0$, so we may assume $x_n \neq 0$. By replacing x_i with $x_n^{-1} x_i$ for $i \leq n - 1$, we may further assume that $x_n = 1$. By induction, we are quickly reduced to the case that

$$\mu(\lambda_n) = \max\{|x_i| \mu(\lambda_i) \mid 1 \leq i \leq n - 1\},$$

since the result otherwise follows quickly from the triangle inequality for the norm. Let us assume this equality holds.

Since $F_\infty = A + m_\infty$, we can write $x_i = a_i + r_i$ for some $a_i \in A$ and $r_i \in m_\infty$ for all $1 \leq i \leq n - 1$. We then have

$$\sum_{i=1}^n x_i \lambda_i = \left(\sum_{i=1}^{n-1} r_i \lambda_i \right) + \left(\lambda_n + \sum_{i=1}^{n-1} a_i \lambda_i \right).$$

Note that

$$\mu \left(\sum_{i=1}^{n-1} r_i \lambda_i \right) = \max\{|r_i| \mu(\lambda_i) \mid 1 \leq i \leq n - 1\} < \mu(\lambda_n).$$

By condition (ii), we also have

$$\mu \left(\lambda_n + \sum_{i=1}^{n-1} a_i \lambda_i \right) \geq \mu(\lambda_n),$$

and it follows from these two inequalities that

$$\mu \left(\sum_{i=1}^n x_i \lambda_i \right) = \mu \left(\lambda_n + \sum_{i=1}^{n-1} a_i \lambda_i \right).$$

On other hand, for $1 \leq i \leq n - 1$ such that $a_i \neq 0$, we have $\mu(a_i \lambda_i) = \mu(x_i \lambda_i)$. Hence

$$\mu \left(\lambda_n + \sum_{i=1}^{n-1} a_i \lambda_i \right) \leq \max(\mu(\lambda_n), \max\{|a_i| \mu(\lambda_i) \mid 1 \leq i \leq n - 1\}) = \mu(\lambda_n).$$

Therefore, we have $\mu(\sum_{i=1}^n x_i \lambda_i) = \mu(\lambda_n)$, as desired.

Claim 2. $(\lambda_i)_{1 \leq i \leq n}$ is an A -basis of Λ .

We prove Claim 2. We can write any nonzero $\lambda \in \Lambda$ as

$$\lambda = \left(\sum_{i=1}^{n-1} a_i \lambda_i \right) + b \lambda_n$$

with $a_i \in A$ and $b \in F$. Suppose $b \notin A$, and write $b = a_n + r$ with $a_n \in A$ and $r \in F$ such that $|r| < 1$. Replacing λ by $\lambda - \sum_{i=1}^n a_i \lambda_i$, we may assume that $\lambda = r \lambda_n$. Then $\mu(\lambda) < \mu(\lambda_n)$. This contradicts condition (ii).

The proof of (3) is straightforward. (4) follows from (3). \square

Proposition 3.1.2. *The map $\mathbb{R}_{>0}^n \rightarrow |BT_n|$; $s \mapsto \text{class}(\mu_s)$ induces a bijection*

$$\{s \in \mathbb{R}_{>0}^n \mid s_1 \leq \cdots \leq s_n\} / \mathbb{R}_{>0} \xrightarrow{\cong} PGL_n(A) \setminus |BT_n|.$$

Proof. Let $P = \{s \in \mathbb{R}_{>0}^n \mid s_1 \leq \cdots \leq s_n\} / \mathbb{R}_{>0}$, $Q = PGL_n(A) \setminus |BT_n|$, and let R be the set of all isomorphism classes of pairs (Λ, μ) where Λ is a free A -module of rank n and μ is a homothety class of a norm on the F_∞ -vector space $F_\infty \otimes_A \Lambda$. We have a canonical map $Q \rightarrow R$; $\mu \mapsto (A^n, \mu)$. We have the map $R \rightarrow P$ which sends the class of (Λ, μ) to the class of $(\tilde{\mu}(\lambda_1), \dots, \tilde{\mu}(\lambda_n))$, where $(\lambda_i)_{1 \leq i \leq n}$ is an A -base of Λ satisfying the equivalent conditions in 3.1.1 (1) and $\tilde{\mu}$ is a norm with class μ . This map $R \rightarrow P$ is well defined by 3.1.1 (4). Then as is easily seen, the compositions $P \rightarrow Q \rightarrow R \rightarrow P$ and $Q \rightarrow R \rightarrow P \rightarrow Q$ are the identity maps. \square

3.1.3. Let $d \geq 1$ and let

$$C_d := \{(s_1, \dots, s_{d-1}) \in \mathbb{R}^{d-1} \mid 0 \leq s_1 \leq \cdots \leq s_{d-1}\}.$$

Let \mathcal{V} be a complete valuation ring of height one over $A = \mathbb{F}_q[T]$ and let ϕ be a generalized Drinfeld module over \mathcal{V} of generic rank d . We define

$$c(\phi) \in C_d / \mathbb{R}_{>0}$$

as follows. Let r and (ψ, Λ) be as in 2.3.5 (1), and let $n = d - r$ be the rank of Λ . By 2.8.21, we have a homothety class μ of a norm on $F_\infty \otimes_A \Lambda$. By the proof of 3.1.2, (Λ, μ) determines an element x of $\{s \in \mathbb{R}_{>0}^n \mid s_1 \leq \cdots \leq s_n\} / \mathbb{R}_{>0}$. We define $c(\phi) = (0^{r-1}, x)$. Thus, $c(\phi)$ is the class of $(0^{r-1}, \tilde{\mu}(\lambda_1), \dots, \tilde{\mu}(\lambda_n)) \in C_d$ where $(\lambda_1, \dots, \lambda_n)$ is an A -base of Λ satisfying the equivalent conditions in 3.1.1 (1) and $\tilde{\mu}$ is a norm with class μ .

3.1.4. Let $k \geq 1$ be an integer. Let ${}_d \Sigma^{(k)}$ (denoted also simply by $\Sigma^{(k)}$) be the cone decomposition of C_d defined as follows. Let $I = \{(h, i, j) \in \mathbb{Z}^3 \mid 1 \leq j < i \leq d - 1, 0 \leq h \leq k - 1\}$. For a map $\alpha : I \rightarrow \{\mathbb{R}_{\leq 0}, \{0\}, \mathbb{R}_{\geq 0}\}$, let

$$\sigma(\alpha) = \{s \in \mathbb{R}^{d-1} \mid 0 \leq s_1 \leq \cdots \leq s_{d-1}, q^h s_j - s_i \in \alpha(h, i, j) \text{ for all } (h, i, j) \in I\}.$$

Then $\Sigma^{(k)}$ is the set of the cones $\sigma(\alpha)$ for these maps α .

This is a coarser version of the restriction of the cone decomposition $\{\sigma(S) \mid S \in AP_{d-1}\}$ (2.8.14) to C_d .

Example 3.1.5. Assume $d = 3$.

For integers $h \geq 1$, let

$$\sigma_h = \{(s_1, s_2) \in \mathbb{R}_{\geq 0}^2 \mid q^{h-1}s_1 \leq s_2 \leq q^h s_1\}.$$

This is the cone $\sigma(S_h)$ in 2.8.16 associated to the simplex S_h of AP_2 . For $h \geq 0$, let

$$\sigma^h = \{(s_1, s_2) \in \mathbb{R}_{\geq 0}^2 \mid q^h s_1 \leq s_2\} = (\cup_{h' > h} \sigma_{h'}) \cup \{(0, s) \mid s \geq 0\}.$$

Then for $k \geq 1$, $\Sigma^{(k)}$ consists of σ_h for $1 \leq h \leq k-1$ and σ^{k-1} , and their faces.

The remaining part of this Section 3.2 is about torsion points and cone decompositions. We give statements of our results which will be proved in Section 3.4.

Proposition 3.1.6. *Let \mathcal{V} be a complete valuation ring of height one over $A = \mathbb{F}_q[T]$ with field of fractions K and let ϕ be a generalized Drinfeld module over \mathcal{V} of generic rank d whose line bundle is trivialized. Let N be an element of A which does not belong to \mathbb{F}_q .*

(1) *Let (ψ, Λ) , r and $n = d - r$ be as in 2.3.5, and let e_Λ be the exponential map of Λ . Let $(\beta_i)_{1 \leq i \leq n}$ be a family of N -torsion points of ϕ in \bar{K} . Then the following conditions (i) and (ii) are equivalent.*

(i) $\beta_i = e_\Lambda(\tilde{\lambda}_i)$ for $1 \leq i \leq n$ where $\tilde{\lambda}_i$ are elements of \bar{K} such that $(\lambda_i)_{1 \leq i \leq n}$ with $\lambda_i = \psi(N)\tilde{\lambda}_i$ is an A -base of Λ satisfying the equivalent conditions in 3.1.1 (1).

(ii) For $1 \leq i \leq n$,

$$-v_{\bar{K}}(\beta_i) = \min \left\{ -v_{\bar{K}}(\beta) \mid \beta \in \phi[N] \setminus \left(e_\Lambda(\psi[N]) + \sum_{j=1}^{i-1} \phi(A/NA)\beta_j \right) \right\}.$$

(2) *For given $a_i \in A/NA$ ($1 \leq i \leq n$), $v_{\bar{K}}(\sum_{i=1}^n a_i \beta_i)$ is independent of the choice of $(\beta_i)_i$ as in (1). In particular, $v_{\bar{K}}(\beta_i)$ is independent of the choice of such $(\beta_i)_i$.*

3.1.7. Let \mathcal{V} be a complete valuation ring of height one over $A = \mathbb{F}_q[T]$, and let ϕ be a generalized Drinfeld module over \mathcal{V} of generic rank d . Let N be an element of A which does not belong to \mathbb{F}_q . We define

$$c(\phi, N) \in C_d/\mathbb{R}_{>0}$$

as the class of $(0^{r-1}, (-v_{\bar{K}}(\beta_i))_{1 \leq i \leq n})$, where r and $(\beta_i)_{1 \leq i \leq n}$ are as in 3.1.6.

Theorem 3.1.8. *Let $k \geq 1$. For each $\sigma \in \Sigma^{(k)} = {}_d\Sigma^{(k)}$, there is a unique finitely generated rational subcone σ' of C_d satisfying the following condition (*):*

(*) *For every complete valuation ring \mathcal{V} of height one over $A = \mathbb{F}_q[T]$ and for every generalized Drinfeld module ϕ over \mathcal{V} , and for every $N \in A$ whose degree as a polynomial over \mathbb{F}_q is k , $c(\phi, N) \in \sigma'$ if and only if $c(\phi) \in \sigma$.*

*It is also characterized by the following condition (**).*

(**) *The same condition as (*) except that complete valuation ring of height one in (*) is replaced by complete discrete valuation ring.*

When $\sigma \in \Sigma^{(k)}$ varies, these σ' form a finite rational subdivision $\Sigma_k = {}_d\Sigma_k$ of C_d .

Proposition 3.1.9. *The fan Σ_1 coincides with the fan of all faces of C_d .*

Proposition 3.1.10. *Let k and k' be integers such that $1 \leq k \leq k'$. If τ is a finitely generated rational cone in \mathbb{R}^{d-1} and is a subcone of some element of $\Sigma_{k'}$, there is a unique finitely generated rational cone σ in \mathbb{R}^{d-1} which is a subcone of some element of Σ_k satisfying the following condition:*

For every complete valuation ring \mathcal{V} of height one over $A = \mathbb{F}_q[T]$ and for every generalized Drinfeld module ϕ over \mathcal{V} , and for every $N, N' \in A$ such that the degree of N is k and the degree of N' is k' and such that $N|N'$, $c(\phi, N') \in \tau$ if and only if $c(\phi, N) \in \sigma$. If Σ' is a subdivision of $\Sigma_{k'}$, these σ associated to $\tau \in \Sigma'$ form a cone decomposition of C_d which is a subdivision of Σ_k .

Let $\Sigma_{k,k'}$ be the subdivision of Σ_k associated to the subdivision $\Sigma_{k'}$ itself of $\Sigma_{k'}$. Thus we have a one-to-one correspondence between subdivisions of $\Sigma_{k'}$ and subdivisions of $\Sigma_{k,k'}$ through this correspondence.

3.2 Study of Drinfeld exponential maps

3.2.1. Fix integers $r \geq 1$ and $n \geq 0$. Let $s = (s_i)_{1 \leq i \leq n} \in \mathbb{R}_{>0}^n$. We define a map

$$\epsilon_s^{r,n} : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$$

as follows:

$$\epsilon_s^{r,n}(x) = \sum_{y \in A^n} \max(x - \max\{|y_i|^r s_i \mid 1 \leq i \leq n\}, 0).$$

We have

$$\epsilon_s^{r,n}(x) = x + \sum_{m \in \mathbb{Z}_{\geq 0}^n} \left(q^{\sum_{i=1}^n m^{(i)}} (q-1)^n \max(x - \max\{q^{m^{(i)r}} s_i \mid 1 \leq i \leq n\}, 0) \right).$$

In the case $n = 0$, we have $\epsilon_s^{r,n}(x) = x$, for s the empty tuple.

Relations between this map $\epsilon_s^{r,n}$ and the Drinfeld exponential map are given in 3.2.7 and 3.2.9.

Example 3.2.2. Consider the case $n = 1$. Let $s = s_1 \in \mathbb{R}_{>0}$, $x \in \mathbb{R}_{\geq 0}$, $h \in \mathbb{Z}_{\geq 0}$, and assume $q^{(h-1)r}s \leq x \leq q^{hr}$. Then

$$\epsilon_s^{r,1}(x) = q^h x - (q^{h(r+1)} - 1) \frac{q - 1}{q^{r+1} - 1} s.$$

3.2.3. This is a preparation for the following 3.2.4.

For $n \geq 1$ and an simplex S of AP_n , we defined a finitely generated rational cone $\sigma(S)$ in $\mathbb{R}_{\geq 0}^n$ (2.8.13). We define here a modified version $\sigma_r(S)$ of $\sigma(S)$ for each integer $r \geq 1$ by

$$\sigma_r(S) := \{(x_1^r, \dots, x_n^r) \mid x \in \sigma(S)\}.$$

This $\sigma_r(S)$ is also a finitely generated rational cone in \mathbb{R}^n . In fact, if α is a map $I = \{(h, i, j) \in \mathbb{Z}^3 \mid 1 \leq i < j \leq n\} \rightarrow \{\mathbb{R}_{\leq 0}, \{0\}, \mathbb{R}_{\geq 0}\}$ such that $\sigma(S)$ coincides with the set of all $x \in \mathbb{R}_{\geq 0}^n$ satisfying $q^h x_i - x_j \in \alpha(h, i, j)$ for all $(h, i, j) \in I$ (2.8.14), $\sigma_r(S)$ coincides with the set of $x \in \mathbb{R}_{\geq 0}^n$ satisfying $q^{hr} x_i - x_j \in \alpha(h, i, j)$ for all $(h, i, j) \in I$

Lemma 3.2.4. *Let S be a simplex of AP_{n+1} . Then there is a linear map $l : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ such that $\epsilon_s^{r,n}(x) = l(s_1, \dots, s_n, x)$ for all $s \in \mathbb{R}_{>0}^n$ and $x \in \mathbb{R}_{>0}$ satisfying $(s_1, \dots, s_n, x) \in \sigma_r(S)$.*

This follows from the definition of $\epsilon_s^{r,n}$.

Corollary 3.2.5. *The map $\epsilon_s^{r,n} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a piecewise linear, increasing homeomorphism.*

3.2.6. Let \mathcal{V} , K , and \bar{K} be as in 2.3.3. Let Λ be a subgroup of the additive group \bar{K} such that the set $\{\lambda \in \Lambda \mid v_{\bar{K}}(\lambda) \geq c\}$ is finite for all c .

Let $e_\Lambda : \bar{K} \rightarrow \bar{K}$ be Drinfeld exponential map associated to Λ defined by

$$e_\Lambda(z) = z \prod_{\lambda \in \Lambda \setminus \{0\}} (1 - \lambda^{-1}z).$$

The following 3.2.7 and 3.2.9 give relations between the function $\epsilon_s^{r,n}$ and Drinfeld exponential map.

Proposition 3.2.7. *Let \mathcal{V} be a complete valuation ring of height one over $A = \mathbb{F}_q[T]$. Let $d, r, n = d - r$, ϕ , (ψ, Λ) , and λ_i ($1 \leq i \leq n$) be as in Propositions 2.8.21 and 3.1.1. Let $s_i = -v_{\bar{K}}(\lambda_i)$.*

Let $\alpha \in \bar{K}^\times$, let E_1 be the finite set $\{\lambda \in \Lambda \mid \lambda \neq 0, \lambda \in O_{\bar{K}}\alpha\}$, and let $\beta = \alpha \prod_{\lambda \in E_1} \alpha \lambda^{-1}$. Then $-v_{\bar{K}}(\beta) = \epsilon_s^{r,n}(-v_{\bar{K}}(\alpha))$. We have $\beta^{-1}e(\alpha z) \in O_{\bar{K}}[[z]]$, the coefficients of this formal power series converge to 0, and some coefficient of it is a unit of $O_{\bar{K}}$.

Proof. The first assertion follows from the definition of $\epsilon_s^{r,n}$. Let $E_2 = \Lambda \setminus (E_1 \cup \{0\})$. We have

$$e(\alpha z) = \alpha z \prod_{\lambda \in E_1} (1 - \lambda^{-1}\alpha z) \prod_{\lambda \in E_2} (1 - \lambda^{-1}\alpha z) = \beta z \prod_{\lambda \in E_1} (\alpha^{-1}\lambda - z) \prod_{\lambda \in E_2} (1 - \lambda^{-1}\alpha z).$$

□

3.2.8. For $x \in \bar{K}$, let

$$v_{\bar{K},\Lambda}(x) = \max\{v_{\bar{K}}(x - \lambda) \mid \lambda \in \Lambda\}.$$

Note that our assumption on Λ implies that $\{\lambda \in \Lambda \mid v_{\bar{K}}(x - \lambda) \geq c\}$ is finite for all c as well, so this maximum exists.

The map $v_{\bar{K},\Lambda}$ factors through the projection $\bar{K} \rightarrow \bar{K}/\Lambda$, and we use the same symbol to denote the resulting map.

Proposition 3.2.9. *Let \mathcal{V} , d , r , n , (ψ, Λ) , and s_i be as in 3.2.7. Then for $x \in \bar{K}$, we have*

$$-v_{\bar{K}}(e_\Lambda(x)) = \epsilon_s^{r,n}(-v_{\bar{K},\Lambda}(x)).$$

Proof. Let α be an element of $x + \Lambda$ such that $v_{\bar{K}}(\alpha) = v_{\bar{K},\Lambda}(x)$. In the proof of 3.2.7, let $z = 1$. Then for every $\lambda \in E_1$, $\alpha^{-1}\lambda - 1$ is a unit of $O_{\bar{K}}$. Hence 3.2.9 follows by the proof of 3.2.7. \square

Remark 3.2.10. Our theory of cone decomposition for toroidal embeddings bases on simplices of the Bruhat-Tits building. The key point is the following. By 3.2.4 and 3.2.7, 3.2.9, the poles of the Drinfeld exponential map have a linear property on simplices (3.2.4, 3.2.7, 3.2.9) and hence poles of torsion points of generalized Drinfeld modules have linear properties if we introduce cone decompositions related to simplices.

3.2.11. For integers r and n such that $r \geq 1$, $n \geq 0$, and for $s \in \mathbb{R}_{>0}^n$, define

$$\delta^{r,n}(s_1, \dots, s_n) = \frac{q-1}{q^{r+n}-1} \sum_{i=1}^n q^{n-i} \epsilon_{s_1, \dots, s_{i-1}}^{r, i-1}(s_i).$$

If $n = 0$, this is the zero function.

A relationship between $\delta^{r,n}$ and generalized Drinfeld modules over \mathcal{V} is given in 3.2.24.

We define a modified version $\hat{\epsilon}_s^{r,n}$ of $\epsilon_s^{r,n}$ as

$$\hat{\epsilon}_s^{r,n}(x) = \epsilon_s^{r,n}(x) - \delta^{r,n}(s_1, \dots, s_n).$$

Example 3.2.12. As in 3.2.2, let $n = 1$, $s = s_1 \in \mathbb{R}_{>0}$, $x \in \mathbb{R}_{\geq 0}$, $h \in \mathbb{Z}_{\geq 0}$, and assume $q^{(h-1)r} s \leq x \leq q^{hr}$. Then

$$\delta^{r,1}(s) = \frac{q-1}{q^{r+1}-1} s,$$

and hence

$$\hat{\epsilon}_s^{r,1}(x) = \epsilon_s^{r,1}(x) - \delta^{r,1}(s) = q^h x - q^{h(r+1)} \frac{q-1}{q^{r+1}-1} s.$$

As in 3.2.16 and 3.2.17 below for example, formulas concerning $\epsilon_s^{r,n}$ have simple forms when we use $\hat{\epsilon}_s^{r,n}$ in place of $\epsilon_s^{r,n}$.

The following is the $\hat{\epsilon}$ -version of 3.2.4.

Lemma 3.2.13. *Let S be a simplex of AP_{n+1} . Then there is a linear map $l : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ such that $\hat{\epsilon}_s^{r,n}(x) = l(s_1, \dots, s_n, x)$ for all $s \in \mathbb{R}_{>0}^n$ and $x \in \mathbb{R}_{>0}$ satisfying $(s_1, \dots, s_n, x) \in \sigma_r(S)$.*

Proof. This follows from 3.2.4. □

3.2.14. It follows from the definitions that

$$\delta^{r,n+1}(s_1, \dots, s_{n+1}) = \delta^{r,n}(s_1, \dots, s_n) + \frac{q-1}{q^{r+n+1}-1} \hat{\epsilon}_{s_1, \dots, s_n}^{r,n}(s_{n+1}).$$

We also have the following.

Proposition 3.2.15. *Assume that $s_i \leq s_{i+1}$ for $1 \leq i \leq n$. We have*

$$\delta^{r,n+1}(s_1, \dots, s_{n+1}) = \delta^{r,1}(s_1) + \delta^{r+1,n}(s'),$$

where $s' = (\hat{\epsilon}_{s_1}^{r,1}(s_{i+1}))_{1 \leq i \leq n}$.

Proposition 3.2.16. *Assume that $s_i \leq s_{i+1}$ for $1 \leq i \leq n+m-1$. Then we have*

$$\hat{\epsilon}_s^{r,n+m}(x) = (\hat{\epsilon}_{s'}^{r+m,n} \circ \hat{\epsilon}_t^{r,m})(x),$$

where $t = (s_1, \dots, s_m) \in \mathbb{R}_{>0}^m$ and $s' \in \mathbb{R}_{>0}^n$ with $s'_i = \hat{\epsilon}_t^{r,m}(s_{m+i})$.

We note that Proposition 3.2.16 has the following corollary.

Corollary 3.2.17. *Assume that $s_i \leq s_{i+1}$ for $1 \leq i \leq n-1$. Then we have*

$$\hat{\epsilon}_s^{r,n}(x) = (\hat{\epsilon}_{s'_n}^{r+n-1,1} \circ \dots \circ \hat{\epsilon}_{s'_2}^{r+1,1} \circ \hat{\epsilon}_{s'_1}^{r,1})(x)$$

where $s'_i = \hat{\epsilon}_{s_1, \dots, s_{i-1}}^{r,i-1}(s_i)$ for $1 \leq i \leq n$.

Proposition 3.2.18. *Assume that $s_i \leq s_{i+1}$ for $1 \leq i \leq n-1$ and $s_n \leq x$. Then we have*

$$\hat{\epsilon}_s^{r,n}(x) = q^{r+n} \hat{\epsilon}_s^{r,n}(q^{-r}x).$$

We prove these propositions.

3.2.19. First, the case $n = 1$ of 3.2.18 follows from 3.2.2, as it yields

$$\hat{\epsilon}_s^{r,1}(x) = q^h \left(x - \frac{q-1}{q^{r+1}-1} q^{hr} s \right)$$

for $h = h(x)$ as in 3.2.2, and $h(x) = h(q^{-r}x) + 1$ as $s \leq x$.

3.2.20. We will treat 3.2.15 and the case $m = 1$ of 3.2.16 simultaneously by induction on n . Both are easy to see for $n = 0$. Proposition 3.2.16 follows from the case $m = 1$ of 3.2.16 by induction on m as follows. Assuming the result for m and $s_i \leq s_{i+1}$ for all $1 \leq i \leq n + m$, we have

$$\hat{\epsilon}_s^{r,n+m+1} = \hat{\epsilon}_{s''}^{r+m,n+1} \circ \hat{\epsilon}_{s_1, \dots, s_{m+1}}^{r,m} = \hat{\epsilon}_{s'}^{r+m+1,n} \circ \hat{\epsilon}_{s_1''}^{r+m,1} \circ \hat{\epsilon}_{s_1, \dots, s_{m+1}}^{r,m} = \hat{\epsilon}_{s'}^{r+m+1,n} \circ \hat{\epsilon}_{s_1, \dots, s_m}^{r,m},$$

where $s'' \in \mathbb{R}_{>0}^{n+1}$ with $s_i'' = \hat{\epsilon}_{t'}^{r,m}(s_{m+i})$, and where $s' \in \mathbb{R}_{>0}^n$ with $s_i' = s_{i+1}''$.

3.2.21. We show 3.2.16 and 3.2.15 for $n - 1$ imply 3.2.15 for n . To this end, for $s = (s_i)_{i=1}^{n+1}$, $t = (s_i)_{i=1}^n$, $s' = (s_i')_{i=1}^n$, and $t' = (s_i')_{i=1}^{n-1}$ with $s_i' = \hat{\epsilon}_{s_1}^{r,1}(s_{i+1})$ for $1 \leq i \leq n$, we compute

$$\begin{aligned} \delta^{r,n+1}(s) &= \delta^{r,n}(t) + \frac{q-1}{q^{r+n+1}-1} \hat{\epsilon}_t^{r,n}(s_{n+1}) \\ &= \delta^{r,1}(s_1) + \delta^{r+1,n-1}(t') + \frac{q-1}{q^{r+n+1}-1} \hat{\epsilon}_t^{r,n}(s_{n+1}) \\ &= \delta^{r,1}(s_1) + \delta^{r+1,n}(s') + \frac{q-1}{q^{r+n+1}-1} (\hat{\epsilon}_t^{r,n}(s_{n+1}) - \hat{\epsilon}_{t'}^{r+1,n-1}(\hat{\epsilon}_{s_1}^{r,1}(s_{n+1}))) \\ &= \delta^{r,1}(s_1) + \delta^{r+1,n}(s'). \end{aligned}$$

Here, the first step is 3.2.14, the second step is 3.2.15 for $n - 1$ and, the third step is 3.2.14 again, and the fourth step is 3.2.16 for $n - 1$.

3.2.22. We complete the proof of 3.2.15 and $m = 1$ of 3.2.16 by showing that 3.2.15 for n and the proven case of 3.2.18 imply 3.2.16 for $m = 1$ and n .

Let $s \in \mathbb{R}_{>0}^{n+1}$. Let $y = (y_i)_{1 \leq i \leq n+1} \in A^{n+1}$, and consider $m_y = \max(P_y - Q_y, 0)$, where

$$P_y = \max(x - |y_1|^r s_1, 0), \quad Q_y = \max\left(\max_{2 \leq i \leq n+1} \{|y_i|^r s_i\} - |y_1|^r s_1, 0\right).$$

If $x \leq \max_{2 \leq i \leq n+1} \{|y_i|^r s_i\}$, then $m_y = 0$. If $x \geq \max_{2 \leq i \leq n+1} \{|y_i|^r s_i\}$, then

$$m_y = \begin{cases} \max(x - |y_1|^r s_1, 0) & \text{if } \max_{2 \leq i \leq n+1} \{|y_i|^r s_i\} \leq |y_1|^r s_1, \\ \max(x - \max_{2 \leq i \leq n+1} \{|y_i|^r s_i\}, 0) & \text{otherwise.} \end{cases}$$

Hence $m_y = \max(x - \max_{1 \leq i \leq n+1} \{|y_i|^r s_i\}, 0)$ for all $y \in A^{n+1}$, and

$$\sum_{y \in A^{n+1}} m_y = \epsilon_s^{r,n+1}(x).$$

On the other hand, if we fix y_2, \dots, y_{n+1} with $x \geq \max_{2 \leq i \leq n+1} \{|y_i|^r s_i\}$, the sum of m_y for y with these fixed y_2, \dots, y_{n+1} is equal to

$$\begin{aligned} \sum_{y_1 \in A} m_y &= \max\left(\epsilon_{s_1}^{r,1}(x) - \epsilon_{s_1}^{r,1}\left(\max_{2 \leq i \leq n+1} \{|y_i|^r s_i\}\right), 0\right) \\ &= \max\left(\epsilon_{s_1}^{r,1}(x) - \max_{2 \leq i \leq n+1} \{\epsilon_{s_1}^{r,1}(|y_i|^r s_i)\}, 0\right) \\ &= \max\left(\hat{\epsilon}_{s_1}^{r,1}(x) - \max_{2 \leq i \leq n+1} \{\hat{\epsilon}_{s_1}^{r,1}(|y_i|^r s_i)\}, 0\right). \end{aligned}$$

By the case $n = 1$ of 3.2.18 proved in 3.2.19, we have

$$\max_{2 \leq i \leq n+1} \{\hat{\epsilon}_{s_1}^{r,1}(|y_i|^r s_i)\} = \max_{2 \leq i \leq n+1} \{|y_i|^{r+1} \hat{\epsilon}_{s_1}^{r,1}(s_i)\}$$

unless $q^{-r} s_1 > s_i$ for all i (which does not hold by assumption on s) or $y_2 = \cdots = y_{n+1} = 0$. It follows that the sum of all m_y is also given by

$$\sum_{y \in A^{n+1}} m_y = (\epsilon_{s'}^{r+1,m} \circ \hat{\epsilon}_{s_1}^{r,1})(x) + \delta^{r,1}(s_1),$$

where $s' = (\hat{\epsilon}_{s_1}^{r,1}(s_{i+1}))_{1 \leq i \leq n}$. By comparing with our first calculation of the sum and applying 3.2.15 for n , we obtain

$$\epsilon_s^{r,n+1}(x) = (\epsilon_{s'}^{r+1,n} \circ \hat{\epsilon}_{s_1}^{r,1})(x) + \delta^{r,n+1}(s) - \delta^{r+1,n}(s'),$$

which implies 3.2.16 for $m = 1$ and n .

3.2.23. We prove Proposition 3.2.18 using Proposition 3.2.16 by induction on n . Suppose it holds for n : we prove it for $n + 1$. For $s = (s_i)_{i=1}^{n+1}$, $t = (s_i)_{i=1}^n$, and $s' = \hat{\epsilon}_t^{r,n}(s_{n+1})$, we have

$$\hat{\epsilon}_s^{r,n+1}(x) = \hat{\epsilon}_{s'}^{r+1,n} \circ \hat{\epsilon}_t^{r,n}(x) = \hat{\epsilon}_{s'}^{r+1,n}(q^{r+n} \hat{\epsilon}_t^{r,n}(q^{-r}x)),$$

the first inequality by 3.2.16 and our assumption on the s_i , and the second equality by induction and our assumption on the s_i and x . By induction and the fact that $s_n \leq s_{n+1}$, we have

$$q^{-r-n} \hat{\epsilon}_t^{r,n}(s_{n+1}) = \hat{\epsilon}_t^{r,n}(q^{-r} s_{n+1}) \leq \hat{\epsilon}_t^{r,n}(q^{-r}x),$$

the inequality following from the fact that $s_{n+1} \leq x$, since $\hat{\epsilon}_t^{r,n}$ is increasing. By the equality in the case $n = 1$ proven in 3.2.19, we then have

$$\hat{\epsilon}_{s'}^{r+1,n}(q^{r+n} \hat{\epsilon}_t^{r,n}(q^{-r}x)) = q^{r+n+1} \hat{\epsilon}_{s'}^{r+1,n}(\hat{\epsilon}_t^{r,n}(q^{-r}x)).$$

Proposition 3.2.24. *Let \mathcal{V} be a complete valuation ring of height one over A . Let $d, r, n = d - r$, $\phi, (\psi, \Lambda)$, and λ_i ($1 \leq i \leq n$) be as in 3.1.1. Let $s_i = -v_{\bar{K}}(\lambda_i)$. Let $f = c(T, q^d)$ be the coefficient of $\phi(T)$ of the highest degree. Then*

$$v_K(f) = (q^d - 1)\delta^{r,n}(s_1, \dots, s_n).$$

Proof. Take $z \in \bar{K}$ such that $v_{\bar{K}}(z) \ll 0$, $v_{\bar{K}}(z) = v_{\bar{K},\Lambda}(z)$, and $v_{\bar{K}}(\psi(T)z) = v_{\bar{K},\Lambda}(\psi(T)z)$. Since $v_{\bar{K}}(\psi(T)z) = q^r v_{\bar{K}}(z)$, Propositions 3.2.9 and 3.2.18 imply that

$$\begin{aligned} v_{\bar{K}}(e_{\Lambda}(\psi(T)z)) &= -\epsilon_s^{r,n}(-q^r v_{\bar{K}}(z)) \\ &= -q^d \hat{\epsilon}_s^{r,n}(-v_{\bar{K}}(z)) - \delta^{r,n}(s_1, \dots, s_n) \\ &= q^d v_{\bar{K}}(e_{\Lambda}(z)) + (q^d - 1)\delta^{r,n}(s_1, \dots, s_n). \end{aligned}$$

On the other hand, we have

$$v_{\bar{K}}(e_{\Lambda}(\psi(T)z)) = v_{\bar{K}}(\phi(T)(e_{\Lambda}(z))) = q^d v_{\bar{K}}(e_{\Lambda}(z)) + v_{\bar{K}}(f),$$

the latter statement in that $v_{\bar{K}}(e_{\Lambda}(z))$ can be made arbitrarily negative by making $v_{\bar{K}}(z)$ so. \square

3.3 Cone decompositions and torsion points. 2.

We prove the results 3.1.6, 3.1.8, 3.1.9, 3.1.10 stated in Section 3.1.

In this Section 3.3, \mathcal{V} denotes a complete valuation ring of height one over A .

3.3.1. Let ϕ be a generalized Drinfeld module over \mathcal{V} of generic rank d . Assume that the line bundle of ϕ is trivialized. Let (ψ, Λ) be the corresponding pair of a Drinfeld module of rank r over \mathcal{V} and a $\psi(A)$ -lattice Λ of rank $n = d - r$ in K^{sep} . Let μ be the norm on $F_\infty \otimes_A \Lambda$ attached to (ψ, Λ) by Proposition 2.8.21. This norm satisfies $\mu^r = -v_{\bar{K}}$ on $\Lambda \setminus \{0\}$.

3.3.2. Let

$$\psi(N)^{-1}\Lambda = \{x \in \bar{K} \mid \psi(N)x \in \Lambda\},$$

$$\phi[N] = \{x \in \bar{K} \mid \phi(N)x = 0\} \quad \text{and} \quad \psi[N] = \{x \in \bar{K} \mid \psi(N)x = 0\}.$$

We have

$$e_\Lambda: \psi(N)^{-1}\Lambda/\Lambda \xrightarrow{\sim} \phi[N].$$

We have also an exact sequence

$$0 \rightarrow \psi[N] \xrightarrow{e_\Lambda} \phi[N] \rightarrow \Lambda/\psi(N)\Lambda \rightarrow 0.$$

The map $\phi[N] \rightarrow \Lambda/\psi(N)\Lambda$ sends $\beta \in \phi[N]$ to the class of any $\lambda \in \Lambda$ such that there is a $\lambda' \in \bar{K}$ satisfying $\beta = e_\Lambda(\lambda')$ and $\psi(N)\lambda' = \lambda$.

Proposition 3.3.3. *Let $(\lambda_i)_{1 \leq i \leq n} \in \Lambda^n$ satisfy the equivalent conditions in 3.1.1(1). Let $\lambda \in \bar{K}$ be such that $\psi(N)\lambda \in \Lambda$, and for $1 \leq i \leq n$, let $a_i \in A$ with $|a_i| < |N|$ be such that $\psi(N)\lambda \equiv \sum_{i=1}^n \psi(a_i)\lambda_i \pmod{\psi(N)\Lambda}$. Then*

$$-v_{\bar{K}}(e_\Lambda(\lambda)) = \epsilon_s^{r,n}(|N|^{-r} \max\{|a_i|^r s_i \mid 1 \leq i \leq n\})$$

where $s_i = -v_{\bar{K}}(\lambda_i)$.

Proof. By Proposition 2.8.21, we have

$$|N|^r v_{\bar{K},\Lambda}(\lambda) = v_{\bar{K},\Lambda}(\psi(N)\lambda) = -\mu \left(\sum_{i=1}^n \psi(a_i)\lambda_i \right)^r = -\max\{|a_i|^r s_i \mid 1 \leq i \leq n\}.$$

By this, the result is reduced to a direct consequence of Proposition 3.2.9. \square

3.3.4. We prove 3.1.6.

We prove (1). The implication (i) \Rightarrow (ii) follows from 3.3.3. We prove the implication (ii) \Rightarrow (i). Let $(\beta_i)_i$ be as in (i) with the relation there to a base $(\lambda_i)_i$ of Λ satisfying 3.1.1 (1). Let $(\beta'_i)_i$ be as in (ii). We prove that $(\beta'_i)_i$ is as in (i) with the relation to a base $(\lambda'_i)_i$ of Λ satisfying 3.1.1

(1). For each $0 \leq j \leq n$, we construct a base $(\lambda_i^{(j)})_i$ of Λ satisfying 3.1.1 (1) by induction on j , and we will define $\lambda'_i = \lambda_i^{(n)}$. First, let $\lambda_i^{(0)} = \lambda_i$. Assume $1 \leq j \leq n$. By induction on j , we may assume that $j \geq 1$ and that we have a base $(\lambda_i^{(j-1)})_i$ of Λ satisfying 3.1.1 (1) such that if we define $\beta'_i{}^{(j-1)} := e_\Lambda(\tilde{\lambda}_i^{(j-1)})$ for some $\tilde{\lambda}_i^{(j-1)} \in \bar{K}$ satisfying $\psi(N)\tilde{\lambda}_i^{(j-1)} = \lambda_i^{(j-1)}$, then $\beta'_k = \beta_k^{(j-1)}$ for $1 \leq k < j$. Write $\beta'_j = \xi + \sum_{k=1}^n a_k \beta_k^{(j-1)}$, where $\xi \in e_\Lambda(\psi[N])$ and $a_k \in A/NA$. By 3.3.3, if $k \geq j$ and $a_k \neq 0$, then $a_k \in \mathbb{F}_q$ and $-v_{\bar{K}}(\beta_k^{(j-1)}) = -v_{\bar{K}}(\beta'_j)$. Let s be the largest integer k such that $k \geq j$ and $a_k \neq 0$. Define $(\lambda_i^{(j)})_i$ as follows: $\lambda_j^{(j)} = \sum_{k=j}^s a_k \lambda_k^{(j-1)}$, $\lambda_k^{(j)} = \lambda_k^{(j-1)}$ if $k < j$ or $k > s$, $\lambda_k^{(j)} = \lambda_{k-1}^{(j-1)}$ if $j < k \leq s$. Then $(\lambda_i^{(j)})_i$ is a base of Λ satisfying 3.1.1 (1) and $\beta'_k = e_\Lambda(\tilde{\lambda}_k^{(j)})$ for $1 \leq k \leq j$ for some $\tilde{\lambda}_k^{(j)} \in \bar{K}$ such that $\psi(N)\tilde{\lambda}_k^{(j)} = \lambda_k^{(j)}$.

(2) follows by the condition (i) in (1), by 3.3.3, and by the fact that $(\mu(\lambda_i))_i$ is independent of the choice of $(\lambda_i)_i$.

We give preparations for the proofs of 3.1.8, 3.1.9, and 3.1.10.

3.3.5. Let $k \geq 1$ and let $\sigma \in \Sigma^{(k)}$. We define a map

$$\pi_{k,\sigma} : \sigma \rightarrow \mathbb{R}_{\geq 0}^{d-1}$$

as follows, and will relate it to the maps $\xi_{k'}^d$ for $0 \leq k' \leq k$.

For $1 \leq i \leq d-1$, let

$$\theta(i) = \min\{j \in \mathbb{Z} \mid 1 \leq j \leq i, q^{k-1}s_j \geq s_i \text{ for all } s \in \sigma\}$$

(note that this set of j is not empty because it contains i). For $s = (s_i)_{i=1}^{d-1} \in C_d$, set

$$r = r(s) = \min\{i \mid s_i \neq 0\}$$

if it exists and $r = d$ otherwise, and define

$$\pi_{k,\sigma}(s)_i = \begin{cases} 0 & \text{if } 1 \leq i < r \\ \hat{\epsilon}_{s_1^r, \dots, s_{\theta(i)-1}^r}^{r, \theta(i)-r}(s_i^r) & \text{if } r \leq i < d. \end{cases}$$

Example 3.3.6. Let $d = 4$, and let

$$\sigma = \{(s_1, s_2, s_3) \in \mathbb{R}_{\geq 0}^3 \mid qs_1 \leq s_2 \leq s_3, qs_2 \geq s_3\} \in \Sigma^{(2)}.$$

We have $\theta(1) = 1$, $\theta(2) = \theta(3) = 2$. Let $s \in \sigma$. Then $\pi_{2,\sigma}(s)$ is as follows.

First assume $s_1 > 0$. Take $h, h' \in \mathbb{Z}$ such that $q^{h-1}s_1 \leq s_2 \leq q^h s_1$ and $q^{h'-1}s_1 \leq s_3 \leq q^{h'} s_1$ ($h \geq 2$, h' is h or $h+1$). By 3.2.12, we have

$$\pi_{2,\sigma}(s) = (s_1, q^h s_2 - q^{2h}(q+1)^{-1}s_1, q^{h'} s_3 - q^{2h'}(q+1)^{-1}s_1).$$

If $s_1 = 0$, we have $\pi_{2,\sigma}(s) = (0, s_2^2, s_3^2)$.

Proposition 3.3.7. *Let k and σ be as in 3.3.5 and let $0 \leq k' \leq k$. Let i, i' be integers such that $\theta(i) - 1 \leq i' \leq i \leq d - 1$. Let $\pi_{\sigma, k', i, i'} : \sigma \rightarrow \mathbb{R}_{\geq 0}$ be the map defined as follows. For $s \in \sigma$, let $r = r(s)$ be as in 3.3.5 and let*

$$\pi_{k', \sigma, i, i'}(s) = \begin{cases} 0 & \text{if } 1 \leq i < r \\ \hat{\epsilon}_{s_r^r, \dots, s_{i'}^r}^{r, i'+1-r}(q^{-k'r} s_i^r) & \text{if } r \leq i < d. \end{cases}$$

(Note that $\pi_{k, \sigma}(s)_i = \pi_{\sigma, 0, i, \theta(i)-1}(s)$.) Then there is a linear map $l_{k', i, i'} : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ such that

$$\pi_{\sigma, k', i, i'}(s) = l_{k', i, i'} \pi_{k, \sigma}(s)$$

for every $s \in \sigma$.

Proof. Let $b = (s_r^r, \dots, s_{\theta(i)-1}^r)$. For an integer j such that $\theta(i) \leq j \leq i$, let $t_j = \hat{\epsilon}_b^{r, \theta(i)-r}(s_j^r)$. Then $t_i = \pi_{k, \sigma}(s)_i$ by 3.2.18. By 3.2.16,

$$\pi_{\sigma, k', i, i'}(s) = \hat{\epsilon}_{t_{\theta(i)}, \dots, t_{i'}}^{\theta(i), i'+1-\theta(i)}(q^{-k'\theta(i)} t_i).$$

By 3.2.18, $(t_{\theta(i)}, \dots, t_{i'}, q^{-k'\theta(i)} t_i)$ ranges in $\sigma_r(S)$ for some simplex S of $AP_{i'-\theta(i)+2}$ (3.2.3). Hence by 3.2.13, the right hand side is a linear function of $t_{\theta(i)}, \dots, t_{i'}$ and t_i . Recall that $t_i = \pi_{k, \sigma}(s)_i$. By induction on i , for $\theta(i) \leq j < i$, t_j is a linear function of $\pi_{k, \sigma}(s)_{j'}$ for $j' < i$. \square

Proposition 3.3.8. *The map $\pi_{k, \sigma} : \sigma \rightarrow \mathbb{R}_{\geq 0}^{d-1}$ is injective and the image coincides with the cone consisting of all $x \in C_d$ satisfying the following condition (*). Assume σ is defined by $q^h s_j - s_i \in \alpha(h, i, j)$, $\alpha(h, i, j) \in \{\mathbb{R}_{\leq 0}, \{0\}, \mathbb{R}_{\geq 0}\}$ ($1 \leq j < i, 0 \leq h \leq k - 1$).*

For each i, j such that $1 \leq j < i \leq d - 1$ and $\theta(i) - 1 \leq j$, choose an integer $c(i, j)$ such that $\theta(i) - 1 \leq c(i, j) \leq j$. (For example, we can take $c(i, j) = j$.)

() $q^{h(c(i, j)+1)} l_{0, j, c(i, j)}(x) - l_{0, i, c(i, j)}(x) \in \alpha(h, j, i)$ for all (i, j) such that $1 \leq j < i \leq d - 1$.*

*(Here $l_{0, *, *}$ is as in 3.3.7. $l_{0, *, *}$ was not determined uniquely. We can make any choices.)*

Proof. We prove this by induction on d . We may assume $d \geq 2$. Let σ' be the subcone of $\mathbb{R}_{\geq 0}^{d-2}$ consisting of all elements s such that $q^h s_j - s_i \in \alpha(h, i, j)$ for all h, i, j such that $1 \leq j < i \leq d - 2$, $0 \leq h \leq k - 1$. By induction on d , we have

(1) $\pi_{k, \sigma'}(\sigma') = \{a \in \mathbb{R}_{\geq 0}^{d-2} \mid q^{h(c(i, j)+1)} a_j - a_i \in \alpha(h, i, j) \text{ (} 1 \leq j < i \leq d - 2, 0 \leq h \leq k - 1)\}$.

We have

$$\sigma = \{s \in \mathbb{R}_{\geq 0}^{d-1} \mid (s_1, \dots, s_{d-2}) \in \sigma', q^h s_j - s_{d-1} \in \alpha(h, d-1, j) \text{ (} 1 \leq j < d-1, 0 \leq h \leq k-1)\}$$

$$= \{s \in \mathbb{R}_{\geq 0}^{d-1} \mid (s_1, \dots, s_{d-2}) \in \sigma', q^{h(c(i, j)+1)} \pi_{\sigma, 0, j, c(d-1, j)}(s) - \pi_{\sigma, 0, d-1, c(d-1, j)}(s) \in \alpha(h, d-1, j)\}$$

$(1 \leq j < d-1, 0 \leq h \leq k-1)$ by 3.2.18.

Hence

(2) $\pi_{k,\sigma}(\sigma) = \{a \in \mathbb{R}_{\geq 0}^{d-1} \mid (a_1, \dots, a_{d-2}) \in \pi_{k,\sigma'}(\sigma'), q^{h(c(i,j)+1)} l_{k,j,c(d-1,j)}(a) - l_{k,d-1,c(d-1,j)}(a) \in \alpha(h, d-1, j) \ (1 \leq j < d-1, 0 \leq h \leq k-1)\}$.

3.3.8 follows from (1) and (2). \square

Example 3.3.9. Let σ be as in 3.3.6. Take $c(2, 1) = c(3, 1) = c(3, 2) = 1$. Since $\pi_{\sigma,0,i,1}(s) = \pi_{2,\sigma}(s)_i$ for $i = 2, 3$ and $\pi_{\sigma,0,1,1}(s) = q(q+1)^{-1}s_1 = q(q+1)^{-1}\pi_{2,\sigma}(s)_1$, the condition (*) in 3.3.8 can be written as

$$(*) \ q^3(q+1)^{-1}x_1 \leq x_2, x_2 \leq x_3, q^2x_2 \geq x_3.$$

Hence we have

$$\pi_{2,\sigma}(\sigma) = \{(a_1, a_2, a_3) \in \mathbb{R}_{\geq 0}^3 \mid q^3(q+1)^{-1}a_1 \leq a_2 \leq a_3, q^2a_2 \geq a_3\}.$$

Lemma 3.3.10. Let $1 \leq i \leq d-1$ and let $\tilde{\delta}_i : C_d \rightarrow \mathbb{R}$ be the map defined as follows. For $s \in C_d$, let

$$\tilde{\delta}_i(s) = \begin{cases} 0 & \text{if } 1 \leq i < r \\ \delta^{r,i+1-r}(s_r^r, \dots, s_i^r) & \text{if } r \leq i < d. \end{cases}$$

Let $k \geq 0$ and $\sigma \in \Sigma^{(k)}$. Then there is a linear map $l : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ such that

$$\tilde{\delta}_i(s) = l\pi_{k,\sigma}(s)$$

for every $s \in \sigma$.

Proof. For $i \geq 1$, we have

$$\tilde{\delta}_i(s) = \tilde{\delta}_{i-1}(s) + \frac{q-1}{q^{i+1}-1} \pi_{\sigma,0,i,i-1}(s).$$

Hence 3.3.10 follows from 3.3.7 by induction on i . \square

3.3.11. For $k \geq 0$, let

$$\xi_k^d : C_d \rightarrow C_d$$

be the map defined as follows. For $s = (s_i)_{i=1}^{d-1} \in C_d$, let $r = r(s)$ be as in 3.3.5 and let

$$\xi_k^d(s)_i = \begin{cases} 0 & \text{if } 1 \leq i < r \\ \epsilon_{s_r^r, \dots, s_{d-1}^r}^{r,d-r} (q^{-kr} s_i^r) & \text{if } r \leq i < d. \end{cases}$$

Since $\epsilon_{s_r^r, \dots, s_{d-1}^r}^{r,d-r}$ is an increasing bijection $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, the map ξ_k^d is easily seen to provide a bijection $C_d \rightarrow C_d$.

Proposition 3.3.12. *Let r , ϕ and (ψ, Λ) be as in Proposition 2.3.5 (1), let $n = d - r$ be the rank of Λ , and let e_Λ be the exponential map of Λ . Let $(\lambda_i)_{1 \leq i \leq n}$ be as in 3.1.1. Let N be an element of A which does not belong to \mathbb{F}_q , let $\tilde{\lambda}_i$ be an element of \tilde{K} such that $\psi(N)\tilde{\lambda}_i = \lambda_i$, let $\beta_i := e_\Lambda(\tilde{\lambda}_i)$, and let a be a non-zero element of A such that $|a| < |N|$. Then we have*

$$\xi_{k-k'}^d(0^{r-1}, \mu(\lambda_1), \dots, \mu(\lambda_n))_{i+r-1} = -v_{\tilde{K}}(a\beta_i)$$

($a\beta_i$ means $\phi(a)\beta_i$), where μ is the induced norm, k is the degree of N , and k' is the degree of a , and $0^{r-1} = (0, \dots, 0)$ ($r - 1$ times).

In particular (take $a = 1$), we have

$$\xi_k^d(0^{r-1}, \mu(\lambda_1), \dots, \mu(\lambda_n)) = (0^{r-1}, -v_{\tilde{K}}(\beta_1), \dots, -v_{\tilde{K}}(\beta_n)).$$

Proof. Set $s_i = -v_{\tilde{K}}(\lambda_i)$ for $1 \leq i \leq n$ and $s = (s_i)_i$. By definition of μ , we have $s_i = \mu(\lambda_i)^r$. Thus

$$\xi_{k-k'}^d(0^{r-1}, \mu(\lambda_1), \dots, \mu(\lambda_n))_i = \epsilon_s^{r,n}(q^{-(k-k')r} s_i).$$

By Proposition 3.3.3, we have $\epsilon_s^{r,n}(q^{-(k-k')r} s_i) = -v_{\tilde{K}}(a\beta_i)$. \square

Proposition 3.3.13. *Let k and σ be as in 3.3.7 and let $0 \leq k' \leq k$. Then there is a linear map $l_{k'} : \mathbb{R}^{d-1} \rightarrow \mathbb{R}^{d-1}$ such that*

$$\xi_{k'}^d(s) = l_{k'}\pi_{k,\sigma}(s)$$

for every $s \in \sigma$.

Proof. By definition of the maps ϵ in 3.2.1, the first $r - 1$ coordinates of $\xi_k^d(s)$ are zero, and since $s \in C_d$, we have equalities

$$\xi_{k'}^d(s)_i = \epsilon_{s_r^r, \dots, s_i^r}^{r, i+1-r}(q^{-k'r} s_i^r)$$

for all i with $r \leq i \leq d - 1$. Hence

$$\xi_{k'}^d(s)_i = \pi_{\sigma, k', i, i}(s) - \tilde{\delta}_i(s)$$

for all i . Hence 3.3.13 follows from 3.3.7 and 3.3.10. \square

Example 3.3.14. Let σ be as in 3.3.6.

We compute the maps ξ_1^4 and ξ_2^4 restricted to σ . Let $s \in \sigma$.

First assume $s_1 > 0$ take $h, h' \in \mathbb{Z}$ as in 3.3.6. Then by 3.2.12, for $k = 1, 2$, we have

$$\xi_k^4(s_1, s_2, s_3) = (q^{-k} s_1, q^{h-2k} s_2 - (q^{2h-2k} - 1)(q+1)^{-1} s_1, q^{h'-2k} s_3 - (q^{2h'-2k} - 1)(q+1)^{-1} s_1).$$

Next assume $s_1 = 0$. Then

$$\xi_1^4(s) = (0, q^{-2} s_2^2, q^{-2} s_3^2), \quad \xi_2^4(s) = (0, q^{-4} s_2^2, q^{-4} s_3^2).$$

Compare this with the computation of $\pi_{2,\sigma}$ in 3.3.6. We have the linear maps l_k for $k = 1, 2$ such that $\xi_k^4 = l_k \circ \pi_{2,\sigma}$ on σ given by

$$l_k(a_1, a_2, a_3) = (q^{-k} a_1, q^{-2k} a_2 + (q+1)^{-1} a_1, q^{-2k} a_3 + (q+1)^{-1} a_1).$$

Lemma 3.3.15. *Let $s \in C_d \cap \mathbb{Q}^{d-1}$, $1 \leq r \leq d-1$, and assume $s_i = 0$ for $1 \leq i \leq r-1$ and $s_r > 0$. Then there are a complete discrete valuation ring \mathcal{V} and a generalized Drinfeld module ϕ over \mathcal{V} of generic rank d such that the rank of $\phi \bmod m_{\mathcal{V}}$ is r and such that the class $c(\phi)$ in $C_d/\mathbb{R}_{>0}$ coincides with the class of s .*

Proof. Take a discrete valuation ring \mathcal{V}' over A and a Drinfeld module over \mathcal{V}' of rank r . Let $n = d - r$, and let \mathcal{V} be the completion of the local ring of the polynomial ring $\mathcal{V}'[u_1, \dots, u_n]$ in n variables at the maximal ideal generated by $m_{\mathcal{V}'}$. Let K be the field of fractions of \mathcal{V} . Take an integer $c \geq 1$ such that $cs_i \in \mathbb{Z}$ for all $1 \leq i \leq d-1$, take elements λ'_i ($1 \leq i \leq n$) of \mathcal{V}' such that $\text{ord}_K(\lambda'_i) = -cs_{i-r+1}$, let $\lambda_i = \lambda'_i u_i$, and let $\Lambda = \sum_{i=1}^n \psi(A)\lambda_i$. Then Λ is a $\psi(A)$ -lattice in K^{sep} of rank n in the sense of 2.3.3. Let ϕ be the generalized Drinfeld module of generic rank d corresponding to (ψ, Λ) in the correspondence 2.3.5. Then $c(\phi)$ is the class of s . \square

3.3.16. Theorem 3.1.8 follows from 3.3.8 and 3.3.13, and 3.3.15. The cone in Σ_k corresponding to $\sigma \in \Sigma^{(k)}$ is $\xi_k(\sigma)$. By 3.3.15, the correspondence $\sigma \mapsto \sigma'$ is characterized by the condition (***) in 3.1.8 because the map ξ_k is continuous on the subset $\{s \in C_d \mid s_i = 0 \text{ for } 1 \leq i \leq r-1, s_r > 0\}$ of C_d for each r .

3.3.17. We prove 3.1.9. By 3.1.8, the bijection $\xi_1^d : C_d \rightarrow C_d$ induces a bijection $\Sigma^{(1)} \rightarrow \Sigma_1 : \sigma \mapsto \xi_1^d(\sigma)$. The fan $\Sigma^{(1)}$ is the fan of all faces of C_d . A face of C_d has the form $\{s \in \mathbb{R}^{d-1} \mid 0 \leq s_1 \leq \dots \leq s_{d-1}, s_i = s_j \text{ if } (i, j) \in I\}$ for some subset I of $\{1, \dots, d-1\}^2$. For a face σ of C_d , since $\xi_1^d(\sigma) \subset \xi_1^d(C_d) = C_d \in \Sigma_1$, $\xi_1^d(\sigma) \in \Sigma_1$ must be a face of C_d . We have easily $\sigma \subset \xi_1^d(\sigma)$ but $\xi_1^d(\sigma) \neq \tau$ for every face τ of C_d such that $\sigma \subsetneq \tau$. Hence $\xi_1^d(\sigma) = \sigma$.

Remark 3.3.18. As in Example 3.3.22 below, if $d \geq 3$, the map ξ_k^d is not continuous. Furthermore, this map ξ_k^d involves the operator $x \mapsto x^r$ for $r \geq 2$, so it is far from being a linear map. Hence the fact the images of the cones $\sigma \in \Sigma^{(k)}$ under ξ_k^d are still cones is a highly nontrivial fact.

Example 3.3.19. Let σ be as in 3.3.6.

The cones $\xi_1^4(\sigma) \in \Sigma_{1,2}$ and $\xi_2^4(\sigma) \in \Sigma_2$ are given by

$$\xi_k^4(\sigma) = \{(a_1, a_2, a_3) \in \mathbb{R}_{\geq 0}^3 \mid qb_1 \leq b_2 \leq b_3, q^2 b_2 - b_3 - q^k(q-1)b_1 \geq 0\}$$

for $k = 1, 2$.

Proposition 3.3.20. *Let k and k' be positive integers. Define*

$$\xi_{k,k'}^d = \xi_{k'}^d \circ (\xi_k^d)^{-1} : C_d \rightarrow C_d.$$

- (1) *The map $\xi_{k,k'}^d : C_d \rightarrow C_d$ is a homeomorphism.*
- (2) *For each $\sigma \in \Sigma_{k,k'}$, there exists a bijective linear map $l : \mathbb{R}^{d-1} \rightarrow \mathbb{R}^{d-1}$ such that the restrictions of l and $\xi_{k,k'}^d$ to σ coincide.*

Proof. This follows from 3.3.13. □

3.3.21. Proposition 3.1.10 follows from 3.3.20 by 3.3.15. For a subcone σ of an element of $\Sigma_{k',k}$ ($0 \leq k \leq k'$), the corresponding cone in $\Sigma_{k'}$ is $\xi_{k,k'}(\sigma)$.

Example 3.3.22. Assume $d = 3$.

(1) Assume $s_1 > 0$. Let h be the integer such that $q^{h-1}s_1 \leq s_2 \leq q^h s_1$. Then

$$\xi_k^3(s_1, s_2) = \begin{cases} (q^{-k}s_1, q^{h-2k}s_2 - (q^{2(h-k)} - 1)(q+1)^{-1}s_1) & \text{if } h \geq k, \\ (q^{-k}s_1, q^{-k}s_2) & \text{if } h \leq k. \end{cases}$$

On the other hand, we have $\xi_k^3(0, s) = (0, q^{-2k}s^2)$.

In particular, ξ_k^3 is not continuous. In fact, when an integer h tends to ∞ ,

$$\xi_k^3(q^{-h}, 1) = (q^{-k-h}, (q^{h+1-2k} + q^{-h})(q+1)^{-1}),$$

and this tends to $(0, \infty)$ and does not converge to $\xi_{3,k}(0, 1) = (0, q^{-2k})$.

(2) Let $1 \leq k \leq k'$. Then $\Sigma_{k,k'}$ consists of

$$\xi_k^3(\sigma_h) = \{(s_1, s_2) \in \mathbb{R}_{\geq 0}^2 \mid q^{h-1}s_1 \leq s_2 \leq q^h s_1\}$$

for $1 \leq h \leq k$,

$$\xi_k^3(\sigma_h) = \{(s_1, s_2) \in \mathbb{R}_{\geq 0}^2 \mid (q^{2h-1-k} + q^k)s_1 \leq (q+1)s_2 \leq (q^{2h+1-k} + q^k)s_1\}$$

for $k < h \leq k' - 1$,

$$\xi_k^3(\sigma^{k'-1}) = \{(s_1, s_2) \in \mathbb{R}_{\geq 0}^2 \mid (q^{2k'-1-k} + q^k)s_1 \leq (q+1)s_2\},$$

and their faces. Here σ_h and $\sigma^{k'-1}$ are as in 3.1.5.

(3) Under the assumption of (3), if σ is either $\xi_k^3(\sigma_h)$ with $h \geq 1$ or $\xi_k^3(\sigma^{k'-1})$, the map $\xi_{k,k'}^3$ on σ coincides with the linear map $(x, y) \mapsto (q^{k-k'}x, q^k(1 - q^{-2k-2k'})(q+1)^{-1}x + q^{2k-2k'}y)$.

3.4 Moduli functors and cone decompositions

Here we prove 1.2.11 in Introduction first, and then consider the moduli functors which appeared in Introduction and related moduli functors.

Proposition 3.4.1. *Let (S, U) be as in 1.2.7, let $((\mathcal{L}, \phi), \iota)$ be a generalized Drinfeld module over (S, U) of rank d with level N structure, and let*

$$\iota : \left(\frac{1}{N}A/A\right)^d \rightarrow \overline{\mathcal{L}}$$

be the map induced by $\iota : \left(\frac{1}{N}A/A\right)^d \rightarrow \mathcal{L}|_U$ on U (2.6.4). Assume that the condition (div) in 2.7.4 is satisfied. Then we have the same statements as (1) and (2) in 1.2.11 concerning $\text{pole}(\iota(a)) \in M_S/\mathcal{O}_S^\times$ for $a \in \left(\frac{1}{N}A/A\right)^d$ also in this situation.

Proof. We may assume that S is of finite type over A and hence is excellent (so we can apply 2.6.5).

We prove (1). Let $s \in S$, let r be the rank of the fiber of the generalized Drinfeld module (\mathcal{L}, ϕ) at s , and let $n = d - r$. Let E_s be as in 2.6.6. By the condition (div), there are elements b_i ($1 \leq i \leq n$) satisfying the following condition (i).

(i) Let $1 \leq i \leq n$. Then $\text{pole}(\iota(b)) \text{pole}(\iota(b_i))^{-1} \in M_S/\mathcal{O}_S^\times$ at s for all elements b of $(\frac{1}{N}A/A)^d$ which do not belong to $E_s + \sum_{j=1}^{i-1} (A/NA)b_j$.

We define elements e_i ($0 \leq i \leq d - 1$) of $(\frac{1}{N}A/A)^d$ as follows. Let $(e_i)_{0 \leq i \leq r-1}$ be a base of the free A/NA -module E_s of rank r (2.6.6). Let $e_i = b_{i-r+1}$ for $r \leq i \leq d - 1$. Then by 2.6.5 and 3.1.6 (1), $(e_i)_{0 \leq i \leq d-1}$ is an A/NA -base of $(\frac{1}{N}A/A)^d$ and satisfies the condition (1.1) of 1.2.11.

(2) is reduced to 3.1.6 (2) by 2.6.5. \square

3.4.2. We prove 1.2.11. This is reduced to the case S is log regular and hence to 3.4.1 by 2.7.5.

3.4.3. We consider the moduli functors

$$\begin{array}{ll} \overline{\mathfrak{M}}_{N,\Sigma}^d, \overline{\mathfrak{M}}_{N,\Sigma}^d, \overline{\mathfrak{M}}_{N,+,\sigma}^d & : \mathcal{C}_{\log} \rightarrow (\text{Sets}) \\ \overline{\mathfrak{M}}_{N,\text{Sa}}^d, \overline{\mathfrak{M}}_{N,\Sigma}^d, \overline{\mathfrak{M}}_{N,+,\sigma}^d & : \mathcal{C}_{\text{nl}} \rightarrow (\text{Sets}) \end{array} \quad \begin{array}{l} (\overline{\mathfrak{M}}_N^d \supset \overline{\mathfrak{M}}_{N,\Sigma}^d, \overline{\mathfrak{M}}_N^d \supset \overline{\mathfrak{M}}_{N,+,\sigma}^d) \\ (\overline{\mathfrak{M}}_{N,\text{Sa}}^d \supset \overline{\mathfrak{M}}_N^d \supset \overline{\mathfrak{M}}_{N,\Sigma}^d, \overline{\mathfrak{M}}_N^d \supset \overline{\mathfrak{M}}_{N,+,\sigma}^d). \end{array}$$

Here Σ is a finite rational cone decomposition of C_d and σ is a finitely generated rational subcone of C_d .

The first two functors $\overline{\mathfrak{M}}_N^d$ and $\overline{\mathfrak{M}}_{N,\Sigma}^d$ on \mathcal{C}_{\log} were defined in 1.3.2, 1.3.4.

For $(S, U) \in \mathcal{C}_{\text{nl}}$, let $\overline{\mathfrak{M}}_{N,\text{Sa}}^d(S, U)$ be the set of all isomorphism classes of generalized Drinfeld modules over (S, U) of rank d with level N structure (1.2.7). Let $\overline{\mathfrak{M}}_N^d(S, U)$ be its subset consisting of those which satisfy (div) (2.7.4).

We define the remaining moduli functors reviewing the definition of $\overline{\mathfrak{M}}_{N,\Sigma}^d$ on \mathcal{C}_{\log} . Both on \mathcal{C}_{\log} and \mathcal{C}_{nl} , $\overline{\mathfrak{M}}_{N,\Sigma}^d$ (resp. $\overline{\mathfrak{M}}_{N,+,\sigma}^d$) is the part of $\overline{\mathfrak{M}}_N^d$ classifying objects such that locally for the standard basis (resp. locally for some $\sigma \in \Sigma$ and for some basis) $(e_i)_{0 \leq i \leq d-1}$ of the free A/NA -module $(\frac{1}{N}A/A)^d$, the following (i) and (ii) are satisfied.

(i) For every $1 \leq i \leq d - 1$ and for every family $(a_j)_{0 \leq j \leq d-1}$ of elements of A/NA such that $a_j \neq 0$ for some $j \geq i$, we have $\text{pole}(\iota(\sum_{j=0}^{d-1} a_j e_j)) \text{pole}(\iota(e_i))^{-1} \in M_S/\mathcal{O}_S^\times$ in $M_S^{\text{gp}}/\mathcal{O}_S^\times$.

(ii) $(\text{pole}(\iota(e_i))_{1 \leq i \leq d-1} \in [\sigma](S)$ in the sense of 2.5.5, 2.5.6. That is, if $b(i) \in \mathbb{Z}$ ($1 \leq i \leq d-1$) and if $\sum_{i=1}^{d-1} b(i)s_i \geq 0$ for all $s \in \sigma$, then we have $\prod_{i=1}^{d-1} \text{pole}(\iota(e_i))^{b(i)} \in M_S/\mathcal{O}_S^\times$ in $M_S^{\text{gp}}/\mathcal{O}_S^\times$.

On both \mathcal{C}_{\log} and \mathcal{C}_{nl} , as a sheaf functor, we have

$$\overline{\mathfrak{M}}_{N,\Sigma}^d = \bigcup_{\sigma, g} \overline{\mathfrak{M}}_{N,+,\sigma}^d$$

where σ ranges over Σ , g ranges over $\text{GL}_d(A/NA)$. For $\sigma \in \Sigma$, on \mathcal{C}_{\log} , $\overline{\mathfrak{M}}_{N,+,\sigma}^d$ is an open subfunctor of $\overline{\mathfrak{M}}_{N,\Sigma}^d$. That is, for an object S of \mathcal{C}_{\log} and for an element of $\overline{\mathfrak{M}}_{N,\Sigma}^d(S)$, the fiber

product of the functors $S \rightarrow \overline{\mathfrak{M}}_{N,\Sigma}^d \leftarrow \overline{\mathfrak{M}}_{N,+,\sigma}^d$ is represented by an open subscheme of S . For an object (S, U) of \mathcal{C}_{nl} and for an element of $\overline{\mathfrak{M}}_{N,\Sigma}^d(S, U)$, the fiber product of the functors $(S, U) \rightarrow \overline{\mathfrak{M}}_{N,+,\sigma}^d \leftarrow \overline{\mathfrak{M}}_{N,+,\sigma}^d$ is represented by (S', U) for some open set S' of S such that $S' \supset U$.

We used the same notation $(\overline{\mathfrak{M}}_N^d, \overline{\mathfrak{M}}_{N,\Sigma}^d$ and $\overline{\mathfrak{M}}_{N,+,\sigma}^d)$ for functors on different categories \mathcal{C}_{log} and \mathcal{C}_{nl} . This is fine as we will see in Theorem 5.3.8 in Section 5.3.

Proposition 3.4.4. *Let k be the degree of N . Both on \mathcal{C}_{log} and on \mathcal{C}_{nl} , we have*

$$\overline{\mathfrak{M}}_N^d = \overline{\mathfrak{M}}_{N,\Sigma_k}^d.$$

Proof. Let S (resp. (S, U)) be an object of \mathcal{C}_{log} (resp. \mathcal{C}_{nl}) and let $((\mathcal{L}, \phi), \iota)$ be an element of $\overline{\mathfrak{M}}_N^d(S)$ (resp. $\overline{\mathfrak{M}}_N^d(S, U)$). Let $s \in S$. Working locally at s , by 1.2.11 (resp. 3.4.1), we have a base $(e_i)_{0 \leq i \leq d-1}$ of $(\frac{1}{N}A/A)^d$ satisfying the condition (1.1) in 1.2.11. As in 3.1.4, let $I = \{(h, i, j) \in \mathbb{Z}^3 \mid 1 \leq j < i \leq d-1, 0 \leq h \leq k-1\}$. We define a map $\alpha : I \rightarrow \{\mathbb{R}_{\leq 0}, \{0\}, \mathbb{R}_{\geq 0}\}$. Let $(h, i, j) \in I$. Take an element a of A such that $|a| = q^h$. We have either $\text{pole}(\iota(e_i)) \text{pole}(\iota(ae_j))^{-1} \in M_S/\mathcal{O}_S^\times$ or $\text{pole}(\iota(ae_j)) \text{pole}(\iota(e_i))^{-1} \in M_S/\mathcal{O}_S^\times$ at s . In the former case, let $\alpha(h, i, j) = \mathbb{R}_{\leq 0}$. Otherwise, let $\alpha(h, i, j) = \mathbb{R}_{\geq 0}$. Define $\tau \in \Sigma^{(k)}$ by $\tau = \{(s_i)_{1 \leq i \leq d-1} \in C_d \mid q^h s_j - s_i \in \alpha(h, i, j) \text{ for all } (h, i, j) \in I\}$ (3.1.4). Let σ be the cone of Σ_k corresponding to τ (3.1.8). Then by 2.6.5, we have $(\text{pole}(\iota(e_i))_{1 \leq i \leq d-1} \in [\sigma](S')$ for some open neighborhood S' of s in S . \square

3.4.5. We describe the plans of Section 4, 5 for our toroidal compactifications. Let k be the degree of the polynomial N .

In Section 4, we consider the formal scheme version of the moduli functor $\overline{\mathfrak{M}}_{N,+,\sigma}^d$. Assuming that σ is contained in some cone in Σ_k , we will prove there that it is represented by a formal scheme which is strongly related to the toric variety $\text{toric}_{\mathbb{F}_p}(\sigma) = \text{toric}_{\mathbb{Z}}(\sigma) \otimes_{\mathbb{Z}} \mathbb{F}_p$ (2.5.6).

In Section 5, we prove that the moduli functor $\overline{\mathfrak{M}}_{N,\Sigma}^d$ on \mathcal{C}_{log} is represented by an fs log scheme $\overline{\mathcal{M}}_{N,\Sigma}^d$ over A which has the properties in Theorem 1.3.5 and Theorem 1.3.6, and that the moduli functor $\overline{\mathfrak{M}}_{N,\Sigma}^d$ on \mathcal{C}_{nl} is represented by $(\overline{\mathcal{M}}_{N,\Sigma}^d, \mathcal{M}_{N,\Sigma}^d)$. In the proofs, the relation with toric varieties proved in Section 4 is used.

4 Tate uniformizations, iterated Tate uniformizations, and formal moduli

In Section 4.1–4.2, we will generalize the correspondence

$$\phi \leftrightarrow (\psi, \Lambda)$$

of Drinfeld for complete discrete valuation fields in 2.3.5 to normal adic spaces. This is the theory of Tate uniformizations for Drinfeld modules. However, as is described in Section 1.4, not like the corresponding theory for abelian varieties, this is not so useful for the toroidal compactifications of the moduli spaces of Drinfeld modules. In Sections 4.3–4.5, we give the moduli theory of log Drinfeld modules on formal schemes using iterated Tate uniformizations, which is useful as the local theory for our toroidal compactifications.

4.1 Construction of the quotient

4.1.1. We begin by setting up the necessarily notation.

Let R be an excellent normal integral domain over A . Let Q be the field of fractions of R , let \bar{Q} be an algebraic closure of Q , let $Q^{\text{sep}} \subset \bar{Q}$ be the separable closure of Q , and let \bar{R} be the integral closure of R in \bar{Q} .

Let I be an ideal of R such that R/I is reduced, and assume that R is I -adically complete. Let \bar{I} be the radical of the ideal $I\bar{R}$ of \bar{R} .

Let ψ be a generalized Drinfeld module (not necessarily a Drinfeld module) over R of generic rank r . We assume that the line bundle of ψ is trivial. Let Λ be an A -submodule of Q^{sep} for the action of A via ψ .

We suppose that the following conditions are satisfied:

- (i) Λ is stable under that action of $\text{Gal}(Q^{\text{sep}}/Q)$.
- (ii) As an A -module, Λ is projective of rank n .
- (iii) For every non-zero element λ of Λ , we have $\lambda^{-1} \in \bar{I}$.
- (iv) For each $n \geq 1$, we have $\lambda^{-1} \in I^n \bar{R}$ for almost all non-zero elements λ of Λ .

We will construct a generalized Drinfeld module ϕ over R of generic rank $d := r + n$ such that $\phi \equiv \psi \pmod{I}$ in 4.1.3–4.1.7. It can be regarded as the quotient of ψ by Λ .

4.1.2. We will use the following cases of this construction:

- (1) ψ is a Drinfeld module of rank r .

This case is useful in Section 4.2 in the direction $(\psi, \Lambda) \mapsto \phi$ of the correspondence in the Tate uniformization. It was considered by K. Fujiwara and R. Pink ([24, page 181]). (The formulation there is slightly different from what is discussed in this Section 4.1.)

- (2) $A = \mathbb{F}_q[T]$ and $n = 1$.

This case is useful for iterated Tate uniformizations in Section 4.5.

4.1.3. Define a formal power series $e(z)$ over R in one variable z by

$$e(z) := z \prod_{\lambda} \left(1 - \frac{z}{\lambda}\right) \in R[[z]],$$

where λ ranges over all non-zero elements of Λ . We have

$$e(z) \equiv z \pmod{I}.$$

Hence $e(z)$ has an inverse function $e^{-1}(z) \in R[[z]]$ satisfying $e^{-1}(z) \equiv z \pmod{I}$.

4.1.4. For $a \in A$, let

$$\phi(a)(z) := (e \circ \psi(a) \circ e^{-1})(z) \in R[[z]].$$

We have $\phi(a)(z) \equiv \psi(a)(z) \pmod{I}$.

4.1.5. The power series $\phi(a)(z)$ is a polynomial of degree $|a|^d$.

4.1.6. We prove 4.1.5. By 2.6.5, we are reduced to the case R is a complete discrete valuation ring. In this case, we can imitate the proof of Drinfeld who proved the case ψ is a Drinfeld module over R ([5]). Let $\psi(a)^{-1}\Lambda = \{x \in \bar{Q} \mid \psi(a)(x) \in \Lambda\}$. By comparing zeros using nonarchimedean analysis over Q , we have

$$e(\psi(a)z) = c \prod_{\beta \in \psi(a)^{-1}\Lambda/\Lambda} (e(z) - e(\beta))^{m(\beta)}$$

for some constant $c \in Q^\times$ and $m(\beta)$ is defined as follows. Take a lifting $\tilde{\beta}$ of β in $\psi(a)^{-1}\Lambda$ and let $\gamma := \psi(a)(\tilde{\beta})$. Then $m(\beta)$ denotes the multiplicity of the root $\tilde{\beta}$ of the polynomial $\psi(a)(x) - \gamma$, which is independent of the choice of $\tilde{\beta}$. Hence $e(\psi(a)(z)) = \varphi(a)e(z)$ is a polynomial of $e(z)$ of degree $|a|^d$.

4.1.7. By 4.1.5, we have constructed a generalized Drinfeld module ϕ over R of generic rank d . such that $\phi \equiv \psi \pmod{I}$.

In the remainder of this section 4.1, we consider the case the lattice Λ is of rank 1 assuming $A = \mathbb{F}_q[T]$.

Proposition 4.1.8. *Let (R, I, Q, r, ψ) be as above. Assume $A = \mathbb{F}_q[T]$.*

Let $f = c(T, q^r) \in R$ be the coefficient of $\psi(T)$ of degree q^r . Let t be a nonzero element of I such that $f^{-q}t^{(q^r-1)(q-1)} \in R$. Then the map

$$A \rightarrow Q, \quad a \mapsto \psi(a)(t^{-1})$$

is injective, and the image $\psi(A)(t^{-1})$ satisfies the condition of Λ in Prop. 4.1.1 with $n = 1$.

To prove this Proposition, we use the following Lemma. For $a, b \in Q^\times$, we write $a \sim b$ if $ab^{-1} \in R^\times$.

Lemma 4.1.9. *Let $a \in A \setminus \{0\}$, and let $f_a = c(a, |a|^r)$. Then $f_a \sim f^b$, where $b = \frac{|a|^r - 1}{q^r - 1}$.*

Proof. Consider the case $a = T^m$ ($m \geq 1$). In this case, we compute f_a by using the equality $\psi(T^m) = \psi(T) \circ \cdots \circ \psi(T)$, where the composition is of m elements. From it, we see that the leading coefficient of $\psi(T^m)$ is that of

$$f_{\tau^r} \cdots f_{\tau^r} = f^{1+q^r+\cdots+q^{(m-1)r}} \tau^{nr} = f^b \tau^{mr}$$

For general $a = \sum_{i=0}^m b_i T^i$ with $b_i \in \mathbb{F}_q$ and $b_m \neq 0$, we have $f_a = b_m f_{T^m} \sim f_{T^m}$. \square

To prove Proposition 4.1.8, it is sufficient to prove the following proposition.

Proposition 4.1.10. *Let the assumptions be as 4.1.8. For $a \in A \setminus \{0\}$, we have $\psi(a)(t^{-1}) \neq 0$. Set $h_a := \psi(a)(t^{-1})^{-1}$. Let m be the degree of the polynomial a . Then we have $h_a \sim t$ if $m = 0$ and $h_a \sim t^{q^{rm}} f^{-b}$ if $m \geq 1$, where $b = \frac{q^{rm}-1}{q^r-1}$. We also have $t^{q^{rm-1}} \mid h_a$ if $m \geq 1$. In particular, $h_a \in I$, and the elements h_a approach 0 in the I -adic topology as $m \rightarrow \infty$.*

4.1.11. We prove Proposition 4.1.10.

We have $\psi(a)(t^{-1}) = \sum_{i=0}^{rm} x_i$ with $x_i := c(a, q^i) t^{-q^i}$. If $m = 0$, then clearly $\psi(a)(t^{-1}) \sim t^{-1}$ and hence $h_a \sim t \in I$. Suppose that $m \geq 1$. Set $g = f^{-(q^{rm}-1)/(q^r-1)} t^{q^{rm}-q^{rm-1}}$. Note that $g \in I$ as

$$g^{q^{(q^r-1)}} = (f^{-q} t^{(q^r-1)(q-1)})^{q^{rm-1}} \cdot t^{(q^r-1)(q-1)} \in I$$

and R/I has no nonzero nilpotent elements. By Lemma 4.1.9, we have that

$$x_{rm} = c(a, q^{rm}) t^{-q^{rm}} \sim f^{(q^{rm}-1)/(q^r-1)} t^{-q^{rm}}.$$

Hence

$$x_{rm}^{-1} = g t^{q^{rm-1}} \in t^{q^{rm-1}} I$$

and x_{rm}^{-1} tends to zero I -adically as $m \rightarrow \infty$. For $i < rm$, since $c(a, q^i) \in R$, it follows that

$$x_i x_{rm}^{-1} \sim c(a, q^i) g t^{q^{rm-1}-q^i} \in I.$$

In particular, we have $\psi(a)(t^{-1}) \sim x_{rm} \neq 0$. Hence $h_a \sim x_{rm}^{-1} \in I$ and $h_a \rightarrow 0$ when $m \rightarrow \infty$.

4.2 Tate uniformizations

The following is a generalization of 2.3.5 to normal adic spaces.

Theorem 4.2.1. *Let S be an adic space which has an open covering each member of which is isomorphic to an open subspace of the adic space $\mathrm{Spa}(R, R)$ associated to the formal scheme $\mathrm{Spf}(R)$ for some excellent ring R complete with respect to an ideal I of definition.*

We assume S is normal (that is, all local rings of S are normal).

Fix integers $d, r \geq 1$ such that $r \leq d$. Let $n = d - r$. Then the following two categories (a) and (b) are equivalent.

(a) The category of Drinfeld modules over \mathcal{O}_S on S of rank d satisfying the following condition
 (i). Let I be the ideal \mathcal{O}_S^+ of definition such that \mathcal{O}_S^+/I is reduced.

(i) This Drinfeld module comes from a generalized Drinfeld module ϕ over \mathcal{O}_S^+ such that $\phi \bmod I$ is a Drinfeld module of rank r over \mathcal{O}_S/I .

(b) The category of pairs (ψ, Λ) where ψ is a Drinfeld module of rank r over \mathcal{O}_S^+ and Λ is an A -submodule of $\mathcal{O}_S \otimes_{\mathcal{O}_S^+} \mathcal{L}$ on the étale site $S_{\text{ét}}$ of S , where \mathcal{L} is the line bundle of ψ and the action of A on $\mathcal{O}_S \otimes_{\mathcal{O}_S^+} \mathcal{L}$ is via ψ (not the usual action), satisfying the following conditions.

(i) Locally on $S_{\text{ét}}$, Λ is isomorphic to the constant sheaf associated to a projective A -module of rank n .

(ii) For every non-zero local section λ of Λ , λ is a local basis of the line bundle $\mathcal{O}_S \otimes_{\mathcal{O}_S^+} \mathcal{L}$ and λ^{-1} is topologically nilpotent.

(iii) For any ideal I of definition of \mathcal{O}_S^+ , locally on $S_{\text{ét}}$, $\lambda^{-1} \in I\mathcal{L}$ for almost all non-zero local section λ of Λ .

4.2.2. The functor (b) \mapsto (a) is given by Section 4.1. (We may assume $S = \text{Spa}(B, B^+)$ with $B^+ = R$ as in 4.1.)

The functor (a) \rightarrow (b) is $\phi \mapsto (\psi, \text{Ker}(e))$ where (ψ, e) is as in 2.3.1. (Note that ϕ in (a) is uniquely determined by 2.2.1.) The crucial problem is to show that $\text{Ker}(e)$ is big enough as is discussed in 4.2.6–4.2.17.

In the proof of Theorem 4.2.1, the theory of simplices of the Bruhat-Tits building plays a key role (4.2.10, 4.2.11).

4.2.3. If $r = d$ (so $n = 0$), the categories (a) and (b) coincide, and hence the theorem is evident. We assume $r < d$ (that is, $n > 0$).

Lemma 4.2.4. (Note that $r < d$ is assumed.) *If the category (a) or the category (b) is non-empty, then the adic space S is Tate (that is, a topologically nilpotent unit exists locally).*

Proof. Assume that the category (a) is not empty. Locally on S , trivialize the invertible \mathcal{O}_S^+ -module \mathcal{L}_ϕ of ϕ . Then for an element a of A which is not in the total constant finite field of F , if we write $\phi(a)(z) = \sum_{i=1}^m c_i z^i$ ($c_i \in \mathcal{O}_S^+$) with $m = |a|_\infty^d$, c_m is a topologically nilpotent unit of \mathcal{O}_S .

Next assume that the category (b) is not empty. Locally on S , trivialize the invertible \mathcal{O}_S^+ -module \mathcal{L}_ψ of ψ . Then for a non-zero local section λ of Λ , λ^{-1} is a topologically nilpotent unit of \mathcal{O}_S . \square

In the rest of Section 4.2, we assume S is Tate.

4.2.5. We show that we may assume $A = \mathbb{F}_q[T]$.

There is a finite flat ring homomorphism $f: \mathbb{F}_q[T] \rightarrow A$, and via this homomorphism, a Drinfeld A -module of rank d is regarded as a Drinfeld $\mathbb{F}_q[T]$ -module of rank dd' where d' is the degree of f . A generalized Drinfeld A -module is similarly regarded as a generalized Drinfeld $\mathbb{F}_q[T]$ -module. The category (a) (resp. (b)) for (A, d) is equivalent to the category of objects E of the category (a) (resp. (b)) for $(\mathbb{F}_q[T], dd')$ endowed with a ring homomorphism $h: A \rightarrow \text{End}(E)$ over $\mathbb{F}_q[T]$ such that $\text{Lie}(h(a)) = a$ for all $a \in A$.

By this, the proof of 4.2.1 is reduced to the case $A = \mathbb{F}_q[T]$.

In the rest of Section 4.2, we assume $A = \mathbb{F}_q[T]$.

We prove the following Proposition in 4.2.7–4.2.17.

Proposition 4.2.6. *Let ϕ be as in (a) of 4.2.1 and let (ψ, e) be the associated pair in 2.3.1. On the étale site of S , consider the sheaf $\Lambda := \text{Ker}(e)$.*

(1) Λ satisfies the conditions (i)–(iii) on Λ in (b) in 4.2.1.

(2) For every complete discrete valuation field K with valuation ring \mathcal{V} with a morphism $\text{Spa}(K, \mathcal{V}) \rightarrow S$, the kernel of e of the pullback of ϕ to \mathcal{V} in the separable closure of K coincides with the pullback of Λ .

4.2.7. For the proof of 4.2.6, we may assume that T is invertible in \mathcal{O}_S^+ . In fact, S is covered by the open subspaces S' and S'' such that T is invertible on $\mathcal{O}_{S'}$ and $T - 1$ is invertible on $\mathcal{O}_{S''}$. On S'' , we can argue replacing $T - 1$ by T .

So, we assume that T is invertible on S .

Working étale locally on S , we may assume that our Drinfeld module ϕ_S over \mathcal{O}_S on S has a level T structure. We assume this.

We trivialize the invertible \mathcal{O}_S^+ -module $\mathcal{L}_\psi = \mathcal{L}_\phi$ of ψ and ϕ by using a non-zero T -torsion point of ψ which is a base of \mathcal{L}_ϕ .

Lemma 4.2.8. *Let Φ be the group of all T -torsion points of ϕ_S . Let $\Psi \subset \Phi$ be the group of $e(b)$ where b is a T -torsion point of ψ . Then*

(1) $\Psi \setminus \{0\} \subset (\mathcal{O}_S^+)^{\times}$.

(2) If $\beta \in \Phi \setminus \Psi$, $\beta^{-1} \in \mathcal{O}_S^+$ is a topologically nilpotent unit in \mathcal{O}_S .

Proof. This follows from 2.6.6. □

4.2.9. Note that for sections f, g of $\mathcal{O}_S^+ \cap (\mathcal{O}_S)^{\times}$, we have either $f|g$ or $g|f$ in \mathcal{O}_S^+ locally on S . Hence by 4.2.8, working locally on S , we may assume that there is a base $(\gamma_i)_{0 \leq i \leq d-1}$ of the \mathbb{F}_p -vector space Φ such that $\Psi = \sum_{i=0}^{r-1} \mathbb{F}_p \gamma_i$ and such that if we put $\beta_i = \gamma_{i+r-1}$ ($1 \leq i \leq n$), then for $1 \leq i \leq n$, $\beta_i^{-1} | \beta^{-1}$ in \mathcal{O}_S^+ for every $\beta \in \Phi \setminus (\Psi + \sum_{j=1}^{i-1} \mathbb{F}_p \beta_j)$.

By 4.2.8 (2), working locally on S , we may assume that $\beta_n^{-1} | \beta_1^{-c}$ in \mathcal{O}_S^+ for some $c \geq 1$.

We assume these.

Let Σ be the fan of all faces of the cone

$$\{a \in \mathbb{R}^{d-1} \mid 0 \leq a_1 \leq \cdots \leq a_{d-1}, a_i = 0 \text{ for } 1 \leq i \leq r-1, q^c a_r \geq a_{d-1}\}.$$

Then $(\gamma_i)_i$ determines an element of $[\Sigma](S)$ (2.5.14).

Take an integer k such that $q^{k-1} > c^{1/r}$. Let $\Sigma_{1,k} * \Sigma$ be the join of the fans $\Sigma_{1,k}$ and Σ . By 2.5.15, $[\Sigma_{1,k} * \Sigma](S) = [\Sigma](S)$. Hence by localization on S , we may assume that $(\gamma_i)_i$ determines an element of $[\tau_1](S)$ for some cone τ_1 in $\Sigma_{1,k} * \Sigma$. Take $\tau \in \Sigma_{1,k}$ such that $\tau_1 \subset \tau$. We have $\tau = \xi_{1,k}(\sigma)$ for some $\sigma \in \Sigma^{(k)}$ (3.3.11).

Lemma 4.2.10. *Let σ' be the face $\{s \in \sigma \mid s_i = 0 \text{ for } 1 \leq i \leq r-1\}$ of σ . Then the cone $\sigma_n := \{(s_{i+r-1})_{1 \leq i \leq n} \mid s \in \sigma'\}$ in \mathbb{R}^n is the cone associated to some simplex of AP_n (2.8.13).*

Proof. Since $I^m \subset Rt \subset I^n$ for some $m, n \geq 1$ and for some non-zero element t of R , there is a prime ideal \mathfrak{p} of R of height one such that $I \subset \mathfrak{p}$. Let \mathcal{V} be the completion of the local ring $R_{\mathfrak{p}}$, let K be the field of fractions of \mathcal{V} , and consider the pullback under $\text{Spa}(K, \mathcal{V}) \rightarrow S$.

Let s be the element of σ_n such that $\xi_1^d(0^{r-1}, (s_i)_{1 \leq i \leq n}) = (0^{r-1}, (-v_{\bar{K}}(\beta_i))_{1 \leq i \leq n})$. We have $s_i > 0$ for $1 \leq i \leq n$. Let $\epsilon = \epsilon_{s_1^r, \dots, s_n^r}^{r,n}$. By the fact $\beta_n^{-1} | \beta_1^{-c}$ in R , we have $\epsilon(q^{-r} s_n^r) \leq c\epsilon(q^{-r} s_1^r)$. Hence we have

$$q^{-r} s_1^r = \epsilon(q^{-r} s_1^r) \geq c^{-1} \epsilon(q^{-r} s_n^r) \geq c^{-1} q^{-r} s_n^r.$$

Hence $c^{1/r} s_1 \geq s_n$. If σ_n is not associated to a simplex of AP_n , since $\sigma \in \Sigma^{(k)}$, we should have $s_n \geq q^{k-1} s_1$ and hence $c^{1/r} s_1 \geq q^{k-1} s_1$. This is a contradiction because $s_1 > 0$ and $q^{k-1} > c^{1/r}$. \square

Proposition 4.2.6 is reduced to the following

Proposition 4.2.11. *Assume $S = \text{Spa}(B, B^+)$ with $R := B^+$. Assume the T -level structure $(\beta_i)_{1 \leq i \leq d-1}$ gives an element of $[\tau](S)$ for some $\tau \in \Sigma_{1,k}$ for some $k \geq 1$ and assume that the cone $\sigma \in \Sigma^{(k)}$ such that $\tau = \xi_1^d(\sigma)$ satisfies the condition that for $\sigma' := \{s \in \sigma \mid s_i = 0 \text{ for } 1 \leq i \leq r-1\}$ and for $\sigma_n := \{(s_{i+r-1})_{1 \leq i \leq n} \mid s \in \sigma'\} \subset \mathbb{R}^n$, the cone σ_n corresponds to a simplex of AP_n .*

Then the following holds locally on $\text{Spf}(R)$: There is a finite separable extension Q' of the field of fractions Q of R such that the integral closure B' of B in Q' is étale over B and $\Lambda := \{x \in Q' \mid e(x) = 0\}$ has the following properties. Let R' be the integral closure of R in Q .

(1) Λ satisfies the conditions (i)–(iii) on Λ in (b) in 4.2.1. That is: $\check{A}\check{A}$

(i) Λ is a projective A -module of rank n .

(ii) For every non-zero element λ of Λ , λ is a unit of B' and λ^{-1} is topologically nilpotent.

(iii) For each $n \geq 1$, $\lambda^{-1} \in (IR')^n$ for almost all non-zero elements λ of Λ .

(2) For every complete discrete valuation field K and with valuation ring \mathcal{V} and with a morphism $\text{Spa}(K, \mathcal{V}) \rightarrow S$, the kernel of e of the pullback of ϕ to \mathcal{V} in the separable closure of K coincides with the pullback of Λ .

The proof of 4.2.11 is given in 4.2.13–4.2.17.

Lemma 4.2.12. *Let p be a prime number, let R_1 be a normal integral domain over \mathbb{F}_p with field of fractions K , and let R_2 be a normal subring of K such that $R_1 \subset R_2$. Let u_i ($1 \leq i \leq t$) be elements of R_1 which are invertible in R_2 and let $m \geq 1$ be an integer. Then there are a finite separable extension K' of K and elements v_i ($1 \leq i \leq s$) of K' such that if we denote by R'_j ($j = 1, 2$) the integral closure of R_j in K' , then R'_2 is étale over R_2 , $v_i \in (R'_2)^\times$, and $R'_1 v_i^m = R'_1 u_i$ ($1 \leq i \leq t$).*

Proof. Let p be the characteristic of \mathbb{F}_q . Write $m = ap^b$ with integers $a \geq 1$ and $b \geq 0$ where a is not divisible by p . Take $w_i \in Q^{\text{sep}}$ ($1 \leq i \leq t$) such that $w_i^a = u_i$ and take $v_i \in (Q^{\text{sep}})^\times$ ($1 \leq i \leq t$) such that $(v_i^{-1})^{p^b} - v_i^{-1} = w_i^{-1}$. Let $Q' = Q(v_1, \dots, v_t)$. Then $R'_2 = R_2[v_1, \dots, v_t]$ and R'_2 is étale over R_2 . It remains to prove that for every additive valuation $v : K' \rightarrow \Gamma \cup \{\infty\}$ for a totally ordered abelian group Γ such that $v(x) \geq 0$ for all $x \in R_1$, we have $mv(v_i) = u_i$. In fact, $av(w_i) = v(u_i)$ evidently. If $v(v_i) > 0$, $p^b v(v_i) = v(w_i)$ as is easily seen. If $v(u_i) < 0$, we should have $v(w_i) < 0$. Since $v(w_i) \geq 0$, these prove $p^b v(v_i) = v(w_i)$. \square

4.2.13. Let $\tau, \sigma, \sigma', \sigma_n$ be as 4.2.11 and let $\tau' := \{a \in \tau \mid a_i = 0 \text{ for } 1 \leq i \leq r-1\}$, $\tau_n = \{(a_{i+r-1})_{1 \leq i \leq n} \mid a \in \tau'\} \subset \mathbb{R}^n$. Then by 3.2.4, there is a \mathbb{Q} -linear map $l : \mathbb{Q}^n \rightarrow \mathbb{Q}^n$ such that the bijection $\xi_1^d : \sigma' \rightarrow \tau'$ sends $(0^{r-1}, x) \in \sigma'$ for $x \in \sigma_n$ to $(0^{r-1}, l(x))$. Let $l^{-1} : \mathbb{Q}^n \rightarrow \mathbb{Q}^n$ be a \mathbb{Q} -linear map such that the inverse bijection $\tau' \rightarrow \sigma'$ sends $(0^{r-1}, x) \in \tau'$ for $x \in \tau_n$ to $(0^{r-1}, l^{-1}(x))$. Write $l^{-1}(e_i) = \sum_{j=1}^n a_{ij} e_j$ with $a_{ij} \in \mathbb{Q}$ where $(e_i)_{1 \leq i \leq n}$ denotes the standard base of \mathbb{Q}^n . Take an integer $m \geq 1$ such that $ma_{ij}q^{-r} \in \mathbb{Z}$ for all i, j . By 4.2.12 applied to the case $R_1 = R, R_2 = B, K = Q, p$ the characteristic of $\mathbb{F}_q, t = n$ and $u_i = \beta_i^{-1}$ ($1 \leq i \leq n$), there are a finite separable extension Q' of Q and α'_i ($1 \leq i \leq n$) such that the integral closure B' of B in Q' is étale over B , $\alpha'_i \in (B')^\times$, and the integral closure R' of R in Q' satisfies $R'(\alpha'_i)^m = R' \prod_{j=1}^n (\beta_j)^{ma_{ij}q^{-r}}$ for $1 \leq i \leq n$.

Replacing $\text{Spa}(B, R)$ by $\text{Spa}(B', R')$, we assume that $\alpha'_i \in B$.

These elements α'_i ($1 \leq i \leq n$) have the following property.

Lemma 4.2.14. *Let K be a complete discrete valuation field with valuation ring \mathcal{V} and with a morphism $\text{Spa}(K, \mathcal{V}) \rightarrow S$. Define $s \in \mathbb{R}_{\geq 0}^n$ by $s_i^r = -q^r v_{\bar{K}}(\alpha'_i)$ for $1 \leq i \leq n$. Then $\xi_1(0^{r-1}, s) = (0^{r-1}, (-v_{\bar{K}}(\beta_i))_{1 \leq i \leq n})$.*

Lemma 4.2.15. *Let $1 \leq i \leq n$ and let $f(z) := \beta_i^{-1} e(\alpha'_i z) = \sum_{j=1}^{\infty} c_j z^j$.*

- (1) $c_j \in R$ for all $j \geq 1$.
- (2) $c_j \rightarrow 0$ when $j \rightarrow \infty$.
- (3) The following holds locally on $\text{Spf}(R)$: $c_m \in R^\times$ for some $m \geq 1$.

Proof. By 2.6.5, we may assume that $S = \text{Spa}(K, \mathcal{V})$ where \mathcal{V} is a complete discrete valuation ring over A and K is the field of fractions of \mathcal{V} . Let $E_1 = \{\lambda \in \Lambda \setminus \{0\} \mid \lambda \in O_{\bar{K}} \alpha'_i\}$. This is a

finite set. Let $E_2 = \Lambda \setminus (E_1 \cup \{0\})$. By 4.2.14, $\beta_i \sim \alpha'_i \prod_{\lambda \in E_1} \alpha'_i \lambda^{-1}$ where \sim means that the ratio is a unit of $O_{\bar{K}}$. Thus $\beta_i^{-1} e(\alpha'_i z) \sim z \prod_{\lambda \in E_1} (\lambda(\alpha'_i)^{-1} - z) \prod_{\lambda \in E_2} (1 - \alpha'_i \lambda^{-1} z)$. (See 3.2.7.) This modulo $m_{\bar{K}}$ is $z \prod_{E_1} (\lambda(\alpha'_i)^{-1} - z)$. \square

Lemma 4.2.16. *Let p be a prime number, let R be a normal integral domain over \mathbb{F}_p , and let $f(z) = \sum_{i=0}^{\infty} c_i z^{p^i} \in R[[z]]$ such that $c_0 \neq 0$ and some c_i is a unit. Suppose that there is a multiple $\pi \in R$ of c_0 such that R is π -adically complete and $c_i \rightarrow 0$ in the π -adic topology.*

Let $a \in R^\times$. Then the following holds locally on $\mathrm{Spf}(R)$ (for the π -adic topology): There is a finite separable extension Q' of the field of fractions Q of R having the following properties: if R' denotes the integral closure of R in Q' , there is $u \in (R')^\times$ such that $f(u) = a$ and $R'[c_0^{-1}]$ is étale over $R[c_0^{-1}]$.

Proof. Take $m \geq 0$ such that $c_i \in \pi^3 R$ for all $i > m$. Consider the polynomial

$$h(z) := -a + \sum_{i=0}^m c_i z^{p^i}$$

By 2.6.3 and the fact that some $c_i \in R^\times$, there is an irreducible monic polynomial $P(z)$ over R such that $P(z) | h(z)$. Let v be a root of $P(z)$ and let $Q' = Q(v)$. Since v divides $a \in R^\times$ in R , we have $v \in (R')^\times$. For $i > m$, write $c_i = \pi^3 b_i$ ($b_i \in R$). Then b_i converges to 0 for the π -adic topology.

As R is π -adically complete, there exists $w \in R'$ such that

$$w + \sum_{i=1}^{\infty} c_i (\pi c_0^{-1}) \pi^{2p^i - 3} w^{p^i} = (\pi c_0^{-1}) \sum_{i=m+1}^{\infty} b_i v^{p^i}.$$

(Here, note that $2p^i - 3 > 0$ for all $i \geq 1$.) We have

$$f(\pi^2 w) = \sum_{i=0}^{\infty} c_i \pi^{2p^i} w^{p^i} = c_0 \pi^2 (w + \sum_{i=1}^{\infty} c_i (\pi c_0^{-1}) \pi^{2p^i - 3} w^{p^i}) = \pi^3 \sum_{i=m+1}^{\infty} b_i v^{p^i} = \sum_{i=m+1}^{\infty} c_i v^{p^i}.$$

Let $u = v - \pi^2 w \in R^\times$. We have

$$f(u) = f(v) - f(\pi^2 w) = \sum_{i=0}^m c_i v^{p^i} = a.$$

We prove that $R'[c_0^{-1}]$ is étale over $R[c_0^{-1}]$. By replacing R by the m -adic completion of R by a maximal ideal m of R , we may assume that R is a local ring and complete by the m_R -adic topology for the maximal ideal m_R of R . Let $R\langle z \rangle$ be the m_R -adic completion of the polynomial ring $R[z]$. Let $F = -a + f$ and let n be the degree of the polynomial $F \bmod m_R \in (R/m_R)[z]$. Then $R\langle z \rangle / (F)$ is generated by z^i ($0 \leq i \leq n-1$) as an R -module. Hence we have $z^n \equiv \sum_{i=0}^{n-1} c_i z^i \bmod F$ for some $c_i \in R$. Let P_1 be the polynomial

$z^n - \sum_{i=0}^{n-1} c_i z^i$ and write $P_1 = uF$ with $u \in R\langle z \rangle$. Then $u \bmod m_R \in (R/m_R)[z]$ is an element of $(R/m_R)^\times$, and hence $u \in R\langle z \rangle^\times$. Hence $(F) = (P_1)$ as ideals of $R\langle z \rangle$. Let $P_2 \in R[z]$ be the monic irreducible polynomial of u over Q . Then $P_2 | P_1$. Since $dF/dz = c_0$, we see that dP_2/dz is invertible in $R[c_0^{-1}]\langle z \rangle / (P_2)$. This proves that $R[c_0^{-1}][z] / (P_2)$ is étale over $R[c_0^{-1}]$ and $R[c_0^{-1}] = R[c_0^{-1}][z] / (P_2)$. \square

4.2.17. We complete the proof of 4.2.11 (and hence of 4.2.6).

By 4.2.15 and 4.2.16, there is a finite separable extension Q' of Q such that; B' is étale over B , we find $x \in (R')^\times$ such that $\beta_i^{-1} e(\alpha'_i x) = 1$. Let $\alpha_i := \alpha'_i x$. Then $e(\alpha_i) = \beta_i$.

Let $\Lambda := \sum_{i=1}^n \psi(AT)\alpha_i$. Then $\Lambda \subset \text{Ker}(e)$. For any complete discrete valuation field K with valuation ring \mathcal{V} and with a morphism $\text{Spa}(K, \mathcal{V}) \rightarrow S$, the pullback of Λ in K^{sep} coincides with the kernel of e for the pullback of ϕ to \mathcal{V} . This proves that $\text{Ker}(e)$ is Λ and proves that the pullback property for such $\text{Spec}(K, \mathcal{V}) \rightarrow S$. By reduction to the case of $\text{Spa}(K, \mathcal{V})$ by pullbacks, we see that Λ satisfies the conditions (i)–(iii).

4.2.18. As above, we have the functor $(a) \rightarrow (b)$. The converse functor $(b) \rightarrow (a)$ is obtained by Section 4.1.

They are the converses of each other is reduced to the case of complete discrete valuation rings by 2.6.5.

4.2.19. Let v be a maximal ideal of A and assume that the image of $S \rightarrow \text{Spec}(A)$ induced by $A \rightarrow \Gamma(S, \mathcal{O}_S^+)$ does not contain v . Let A_v be the v -adic completion of A . Then concerning the v -adic Tate modules, we have an exact sequence $0 \rightarrow T_v \psi \xrightarrow{e} T_v \phi \rightarrow A_v \otimes_A \Lambda \rightarrow 0$ on S_{et} .

4.2.20. We compare this Section 4.2 with the corresponding theory of abelian varieties.

Let (B, B^+) be a complete Huber pair such that $R := B^+$ is an excellent normal integral domain and B is contained in the field of fractions of R , and let $S = \text{Spa}(B, B^+)$.

In the theory of degeneration of abelian varieties, we often consider an abelian scheme X over B , a semi-abelian scheme G over R which is an extension of an abelian scheme over R by a torus over R , a subgroup Λ of $G(B)$ which is discrete and which is a free \mathbb{Z} -module of finite rank, and an isomorphism $X_S \cong G_S / \Lambda$, where X_S and G_S are adic spaces over S associated to A and G , respectively. Let ℓ be a prime number which is invertible in R . We have an exact sequence for Tate modules $0 \rightarrow T_\ell G \rightarrow T_\ell X \rightarrow \mathbb{Z}_\ell \otimes_{\mathbb{Z}} \Lambda \rightarrow 0$ on the étale site of S . This is similar to the exact sequence in 4.2.19.

In the case R is a ring over A , the Drinfeld module ψ over R in this Section 4.2 is similar to the above G , Λ in this Section 4.2 is similar to the above Λ of the theory of abelian varieties, the Drinfeld module over ϕ_B over B is similar to the abelian scheme X over B , and the theory to obtain ϕ_S as the quotient of ψ_S by Λ is similar to $A_S = G_S / \Lambda$.

Let Q be the field of fractions of the strict henselization R' of a local ring of R at an open prime ideal of R . In the theory of abelian varieties, if ℓ is a prime number which is invertible in

R , we have an exact sequence $0 \rightarrow T_\ell G \rightarrow T_\ell X \rightarrow \mathbb{Z}_\ell \otimes_{\mathbb{Z}} \Lambda \rightarrow 0$ of $\text{Gal}(Q^{\text{sep}}/Q)$ -modules in which the actions of $\text{Gal}(Q^{\text{sep}}/Q)$ on $T_\ell G$ and on $\mathbb{Z}_\ell \otimes_{\mathbb{Z}} \Lambda$ are trivial. However a big difference is that in the theory of abelian varieties, the lattice Λ appears in $G(B) \subset G(Q^{\text{sep}})$, but in the theory of Drinfeld modules, the lattice Λ appears on $S_{\text{ét}}$ but can not appear in $\mathcal{L}_\psi(Q^{\text{sep}})$, where \mathcal{L}_ψ is the line bundle of ψ . If v is not contained in the image of $\text{Spec}(R) \rightarrow \text{Spec}(A)$, we have a $\text{Gal}(Q^{\text{sep}}/Q)$ -module $T_v \phi$ such that the action of $\text{Gal}(Q^{\text{sep}}/Q)$ on $T_v \psi \subset T_v \phi$ is trivial, but as the examples 5.4.11 and 5.4.12 show, the image of $\text{Gal}(Q^{\text{sep}}/Q)$ in $\text{Aut}(T_v \phi/T_v \psi)$ can be very big and $T_v \phi/T_v \psi$ can not have the form $A_v \otimes_A \Lambda$ such that the action of $\text{Gal}(Q^{\text{sep}}/Q)$ on Λ factors through a finite quotient of $\text{Gal}(Q^{\text{sep}}/Q)$. If we take a complete discrete valuation ring $\mathcal{V} \supset R'$ which dominates R' and if K is the field of fractions of \mathcal{V} , we can identify $T_v \phi/T_v \psi$ with $A_v \otimes_A \Lambda$ where (ψ, Λ) corresponds to ϕ over K and $\Lambda \subset \mathcal{L}_\psi(K^{\text{sep}})$, but this Λ does not appear in $\mathcal{L}_\psi(Q^{\text{sep}}) \subset \mathcal{L}_\psi(K^{\text{sep}})$.

4.3 Log Drinfeld modules in formal geometry

4.3.1. If S is a locally Noetherian formal scheme and S_{adic} is the associated adic space, we have a morphism of locally ringed spaces

$$(1) \quad (S_{\text{adic}}, \mathcal{O}_{S_{\text{adic}}}^+) \rightarrow (S, \mathcal{O}_S).$$

In the case $S = \text{Spf}(R)$, $S_{\text{adic}} = \text{Spa}(R, R)$ and the map $S_{\text{adic}} \rightarrow S$ sends $x \in S_{\text{adic}}$ to the prime ideal $\{f \in R \mid |f(x)| < 1\}$ of R . The ring homomorphism from the inverse image of \mathcal{O}_S to $\mathcal{O}_{S_{\text{adic}}}$ induces the identity map $R = \Gamma(S, \mathcal{O}_S) \rightarrow R = \Gamma(S_{\text{adic}}, \mathcal{O}_{S_{\text{adic}}}^+)$.

Hence if S is over A , a generalized Drinfeld module over S defines a generalized Drinfeld module over $\mathcal{O}_{S_{\text{adic}}}^+$ on S_{adic} by pullback by (1) and hence a generalized Drinfeld module over $\mathcal{O}_{S_{\text{adic}}}$ on S_{adic} .

4.3.2. We consider pairs (S, U) where S is a locally Noetherian formal scheme such that $\mathcal{O}_{S,s}$ is a normal integral domain for every $s \in S$ and U is a dense open subset of S_{adic} .

4.3.3. Let (S, U) be as in 4.3.2 and assume that S is over A .

By a generalized Drinfeld module over (S, U) of rank d with level N structure, we mean a pair $((\mathcal{L}, \phi), \iota)$ of a generalized Drinfeld module over S whose pullback to U via 4.3.1 (1) gives a Drinfeld module over \mathcal{O}_U of rank d and ι is a level N structure of this Drinfeld module over \mathcal{O}_U .

Remark 4.3.4. (1) In 4.3.3, we consider a dense open subset U of S_{adic} , not a dense open subset of S . The reason is as follows. In the case $A = \mathbb{F}_q[[T]]$, a standard situation we consider is a generalized Drinfeld module ϕ over the T -adic completion $\mathbb{F}_q[[[T]]]$ of A which is not a Drinfeld module but which induces a Drinfeld module over $\mathbb{F}_q((T))$. Then ϕ induces a generalized Drinfeld module over the formal scheme $S = \text{Spf}(\mathbb{F}_q[[[T]])$. But S is just a one point set, and we can not have a non-empty open set of S on which the pullback of ϕ is a Drinfeld module. On the other

hand, the pullback of this ϕ on S to $S_{\text{adic}} = \text{Spa}(\mathbb{F}_q[[T]], \mathbb{F}_q[[T]])$ induces a Drinfeld module over \mathcal{O}_U on the dense open set $U = \text{Spa}(\mathbb{F}_q((T)), \mathbb{F}_q[[T]])$ of S_{adic} .

(2) Let (S, U) and $((\mathcal{L}, \phi), \iota)$ be as in 4.3.3. Then via the morphism $(U, \mathcal{O}_U^+) \rightarrow (S, \mathcal{O}_S)$ induced by (1) in 4.3.1, it gives on U a generalized Drinfeld module over \mathcal{O}_U^+ which becomes a Drinfeld module over \mathcal{O}_U . This is a situation considered in Section 4.2 (the present U is the S in Section 4.2).

4.3.5. For (S, U) as in 4.3.2, define the saturated log structure M on S associated to U as $M = \mathcal{O}_S \cap j_*(\mathcal{O}_U^\times)$ in $j_*(\mathcal{O}_U)$, where j is the composition $U \rightarrow S_{\text{adic}} \rightarrow S$ and j_* is for the étale topologies of S and of U . For an object S' of the étale site of S , $M(S')$ is described as follows. We have the induced morphism $S'_{\text{adic}} \rightarrow S_{\text{adic}}$ of adic spaces. Let $U' \subset S'_{\text{adic}}$ be the inverse image of U . Then $M(S')$ is identified with the set of all elements $f \in \mathcal{O}(S')$ whose pullbacks to $\mathcal{O}_{S'_{\text{adic}}}(U')$ are invertible.

For a line bundle \mathcal{L} on S , this log structure M defines $\overline{\mathcal{L}} = \mathcal{L} \cup M^{-1}\mathcal{L}^\times \subset j_*(\mathcal{L}|_U)$ where $\mathcal{L}|_U$ means the invertible \mathcal{O}_U -module on U induced by \mathcal{L} .

Proposition 4.3.6. *Let (S, U) be as in 4.3.3 and let $((\mathcal{L}, \phi), \iota)$ be a generalized Drinfeld module over (S, U) of rank d with level N structure. Then the map $\iota : (\frac{1}{N}A/A)^d \rightarrow \mathcal{L}|_U$ on U extends uniquely to a map $\iota : (\frac{1}{N}A/A)^d \rightarrow \overline{\mathcal{L}}$ on S .*

Proof. This is reduced to 2.6.4 applied to the normal schemes $\text{Spec}(\mathcal{O}_{S,s})$ ($s \in S$). □

4.3.7. Let S be a locally Noetherian formal scheme over A with a saturated log structure. By a log Drinfeld module over S of rank d with level N structure, we mean a pair $((\mathcal{L}, \phi), \iota)$ of a generalized Drinfeld module (\mathcal{L}, ϕ) over S and a map

$$\iota : (\frac{1}{N}A/A)^d \rightarrow \overline{\mathcal{L}}$$

satisfying the following condition (i).

(i) Étale locally on S , there are a log scheme S' over A satisfying the equivalent conditions (i)–(iv) in 1.2.5, a closed subscheme Y of S , a morphism $f: S \rightarrow \hat{S}'$ of log formal schemes over A where \hat{S}' denotes the formal completion of S along Y , a generalized Drinfeld module $((\mathcal{L}', \phi'), \iota')$ over (\hat{S}', U) of rank d with level N structure in the sense of 4.3.3, where U denotes the dense open set of \hat{S}' at which the log structure of \hat{S}' is trivial, and an isomorphism between $((\mathcal{L}, \phi), \iota)$ and the pullback of $((\mathcal{L}', \phi'), \iota')$ (here ι' is regarded as a map $(\frac{1}{N}A/A)^d \rightarrow \overline{\mathcal{L}'}$ (4.3.6)) under f .

This is a formal scheme version of 1.2.8.

4.3.8. Let (S, U) be as in 4.3.3 and let $((\mathcal{L}, \phi), \iota)$ be a generalized Drinfeld module over (S, U) of rank d with level N structure. Let $\iota : (\frac{1}{N}A/A)^d \rightarrow \overline{\mathcal{L}}$ be the induced map. We consider the following formal scheme version of the condition (div) in 2.7.4.

(div) For every $a, b \in (\frac{1}{N}A/A)^d$, we have locally on S either $\text{pole}(a)\text{pole}(b)^{-1} \in M_S/\mathcal{O}_S^\times$ or $\text{pole}(b)\text{pole}(a)^{-1} \in M_S/\mathcal{O}_S^\times$ in $M_S^{\text{gp}}/\mathcal{O}_S^\times$.

Note that again by 2.5.11, we have the same condition (div) if we replace “locally” by “étale locally”.

By 2.7.5 if $\text{Spec}(\mathcal{O}_{S,\bar{s}})$ with the log structure induced by M_S is log regular for every $s \in S$ (for example, if S is the formal completion of a log regular scheme of finite type over A along a closed subscheme) and if U is the dense open subset of S_{adic} consisting of all points at which the log structure is trivial, then the condition (div) satisfied.

Hence we have

Proposition 4.3.9. *Let S be a locally Noetherian formal scheme over A with a saturated log structure, and let $((\mathcal{L}, \phi), \iota)$ be a log Drinfeld module over S of rank d with N level structure. Then:*

For $a, b \in (\frac{1}{N}A/A)^d$, we have locally on S either $\text{pole}(\iota(a))\text{pole}(\iota(b))^{-1} \in M_S/\mathcal{O}_S^\times$ or $\text{pole}(\iota(b))\text{pole}(\iota(a))^{-1} \in M_S/\mathcal{O}_S^\times$ in $M_S^{\text{gp}}/\mathcal{O}_S^\times$.

Proposition 4.3.10. *Assume we are in one of the following situations (i) and (ii).*

(i) Let S be a locally Noetherian formal scheme over A with a saturated log structure, and let $((\mathcal{L}, \phi), \iota)$ be a log Drinfeld module over S of rank d with N level structure.

(ii) Let (S, U) be as in 4.3.3 and let $((\mathcal{L}, \phi), \iota)$ be a generalized Drinfeld module over (S, U) of rank d with level N structure.

Suppose further that either N has at least two prime divisors or N is invertible on S .

Then the automorphism group of $((\mathcal{L}, \phi), \iota)$ is trivial.

Proof. This is reduced to 2.6.10 and 2.7.8. □

4.4 Formal moduli

We assume $A = \mathbb{F}_q[T]$.

Proposition 4.4.1. *Assume we are in one of the following situations (i) and (ii).*

(i) Let S be a locally Noetherian formal scheme over A with a saturated log structure, and let $((\mathcal{L}, \phi), \iota)$ be a log Drinfeld module over S of rank d with N level structure.

(ii) Let S be a formal scheme over A which has an open covering by $\text{Spf}(R)$ with R an excellent normal domain and let U be a dense open subset of the adic space S_{adic} . Let $((\mathcal{L}, \phi), \iota)$ be a generalized Drinfeld module over (S, U) of rank d with level N structure satisfying the divisibility condition (div) in 4.3.8.

Then we have the same statements as (1) and (2) in 1.2.11 concerning $\text{pole}(\iota(a)) \in M_S/\mathcal{O}_S^\times$ for $a \in (\frac{1}{N}A/A)^d$ also in this formal situation.

Proof. This can be proved in the same way as 1.2.11 and 3.4.1 by using the reduction 2.6.5 to the case of complete discrete valuation rings given in Section 3.4. \square

4.4.2. In the case N has at least two prime divisors (resp. N has only one prime divisor), we define the categories $\hat{\mathcal{C}}_{\log}$ and \mathcal{C}_{nl} as follows: Let $\hat{\mathcal{C}}_{\log}$ be the category of locally Noetherian formal schemes S over A (resp. $A[\frac{1}{N}]$) endowed with a saturated log structure M_S on the étale site $S_{\text{ét}}$. Let $\hat{\mathcal{C}}_{\text{nl}}$ be the category of pairs (S, U) where S is a formal scheme over A (resp. $A[\frac{1}{N}]$) which has an open covering by $\text{Spf}(R)$ with R an excellent normal domain and U is a dense open subset of the adic space S_{adic} .

4.4.3. Let $r \geq 1, n \geq 0, d = r + n$.

Let σ be a finitely generated rational subcone of C_d . We define functors

$$\begin{aligned} \overline{\mathfrak{M}}_N^{r,n}, \overline{\mathfrak{M}}_{N,+,\sigma}^{r,n} : \hat{\mathcal{C}}_{\log} &\rightarrow (\text{Sets}) & (\overline{\mathfrak{M}}_N^{r,n} \supset \overline{\mathfrak{M}}_{N,+,\sigma}^{r,n}) \\ \overline{\mathfrak{M}}_N^{r,n}, \overline{\mathfrak{M}}_{N,+,\sigma}^{r,n} : \hat{\mathcal{C}}_{\text{nl}} &\rightarrow (\text{Sets}) & (\overline{\mathfrak{M}}_N^{r,n} \supset \overline{\mathfrak{M}}_{N,+,\sigma}^{r,n}) \end{aligned}$$

as follows

For an object S of \mathcal{C}_{\log} , let $\overline{\mathfrak{M}}_N^{r,n}(S)$ be the set of all isomorphism classes of log Drinfeld modules $((\mathcal{L}, \phi), \iota)$ over S of rank d with level N structure satisfying the following condition (i) below.

For $(S, U) \in \mathcal{C}_{\text{nl}}$, $\overline{\mathfrak{M}}_N^{r,n}(S, U)$ is the set of all isomorphism classes of a generalized Drinfeld module $((\mathcal{L}, \phi), \iota)$ over (S, U) of rank d with level N structure satisfying the divisibility condition (div) (4.3.8) and the following condition (i).

(i) Let $I \subset \mathcal{O}_S$ be the ideal of definition such that \mathcal{O}_S/I is reduced. Then $(\mathcal{L}, \phi) \bmod I$ is a Drinfeld module over \mathcal{O}_S/I of rank r .

Both on $\hat{\mathcal{C}}_{\log}$ and $\hat{\mathcal{C}}_{\text{nl}}$, the subfunctor $\overline{\mathfrak{M}}_{N,+,\sigma}^{r,n}$ of $\overline{\mathfrak{M}}_N^{r,n}$ is defined by putting the conditions on the divisibility of $\text{pole}(\iota(a))$ ($a \in (\frac{1}{N}A/A)^d$) in the same way as the cases of the categories \mathcal{C}_{\log} and \mathcal{C}_{nl} in 3.4.3, respectively.

We use the same notation for functors on different categories \mathcal{C}_{\log} and \mathcal{C}_{nl} , but this is fine as is seen in 4.4.10.

4.4.4. When we consider $((\mathcal{L}, \phi), \iota) \in \overline{\mathfrak{M}}_{N,+,\sigma}^{r,n}$, we trivialize the line bundle \mathcal{L} by $\iota(e_0)$ which is a base of \mathcal{L} .

4.4.5. On the category \mathcal{C}_{\log} (resp. $\hat{\mathcal{C}}_{\text{nl}}$), we have a canonical morphism of functors $((\mathcal{L}, \phi), \iota) \mapsto ((\mathcal{L}, \psi), \iota')$ from $\overline{\mathfrak{M}}_{N,+,\sigma}^{r,n}$ to the functor $S \mapsto \text{Mor}(S, \mathcal{M}_N^r)$ (resp. $(S, U) \mapsto \text{Mor}(S, \mathcal{M}_N^r)$), where Mor is the set of morphisms of locally ringed spaces over A . Here ψ is that of (ψ, e) in 2.3.1 and ι' is defined by $e(\iota'(a_0, \dots, a_{r-1})) = \iota(a_0, \dots, a_{r-1}, 0, \dots, 0)$.

4.4.6. Let k be the degree of the polynomial N . Let r, n be integers such that $r \geq 1, n \geq 0, r + n = d$. Let σ be a finitely generated rational subcone of C_d such that $\sigma \subset \tau$ for some

$\tau \in \Sigma_k = {}_d\Sigma_k$. Let σ' be the face $\{s \in \sigma \mid s_i = 0 \text{ for } 1 \leq i \leq r-1\} \subset \mathbb{R}_{\geq 0}^n$ of σ , and let $\sigma_n = \{(s_{i+r-1})_{1 \leq i \leq n} \mid s \in \sigma'\} \subset \mathbb{R}_{\geq 0}^n$. Hence $\sigma' = \{0\}^{r-1} \times \sigma_n$.

Let p be the characteristic of F and let $\text{toric}_{\mathbb{F}_p}(\sigma_n)$ be the toric variety $\text{toric}_{\mathbb{Z}}(\sigma_n) \otimes_{\mathbb{Z}} \mathbb{F}_p$ over \mathbb{F}_p associated to σ_n (2.5.6). That is, $\text{toric}_{\mathbb{F}_p}(\sigma_n) = \text{Spec}(\mathbb{F}_p[\sigma_n^\vee])$, where $\sigma_n^\vee = \{b \in \mathbb{Z}^n \mid \sum_{i=1}^n b(i)s_i \geq 0 \text{ for all } s \in \sigma_n\}$.

4.4.7. Let the notation be as in 4.4.6.

Let $\overline{\mathcal{M}}_{N,(+,\sigma)}^{r,n}$ be the formal completion of $\mathcal{M}_N^r \times_{\mathbb{F}_p} \text{toric}_{\mathbb{F}_p}(\sigma_n)$ along the following closed subscheme Y . The embedding $\sigma_n \subset \mathbb{R}_{\geq 0}^n$ of cones induces a morphism $\text{toric}_{\mathbb{F}_p}(\sigma_n) \rightarrow \mathbf{A}_{\mathbb{F}_p}^n$ of toric varieties. Let Y be the inverse image of $(0, \dots, 0)$ of $\mathbf{A}_{\mathbb{F}_p}^n$ under $\mathcal{M}_N^r \times_{\mathbb{F}_p} \text{toric}_{\mathbb{F}_p}(\sigma_n) \rightarrow \text{toric}_{\mathbb{F}_p}(\sigma_n) \rightarrow \mathbf{A}_{\mathbb{F}_p}^n$.

Endow $\overline{\mathcal{M}}_{N,(+,\sigma)}^{r,n}$ with the inverse image of the standard log structure of $\text{toric}_{\mathbb{F}_p}(\sigma_n)$. Let W be the dense open set of the adic space associated to $\overline{\mathcal{M}}_{N,(+,\sigma)}^{r,n}$ consisting of all points at which the log structure is trivial.

Thus $\overline{\mathcal{M}}_{N,(+,\sigma)}^{r,n}$ represents the functor

$$\hat{\mathcal{C}}_{\log} \rightarrow (\text{Sets}) ; S \mapsto \{(\psi, \iota, t_1, \dots, t_n)\},$$

where ψ is a Drinfeld module over S with level N structure whose line bundle is trivialized by $\iota(e_0)$ and $t_i \in \Gamma(S, M_S)$ satisfying the following conditions.

- (1) The image of t_i in \mathcal{O}_S is topologically nilpotent.
- (2) The element $(t_i \bmod \mathcal{O}_S^\times)_{1 \leq i \leq n}$ of $\Gamma(S, (M_S^{\text{gp}}/\mathcal{O}_S^\times)^n)$ belongs to $[\sigma_n](S)$. That is, $\prod_{i=1}^n t_i^{b(i)} \in M_S$ for every $b \in \mathbb{Z}^n$ such that $\sum_{i=1}^n b(i)s_i \geq 0$ for all $s \in \sigma_n$.

The above W is the open set of the adic space associated to $\overline{\mathcal{M}}_{N,(+,\sigma)}^{r,n}$ consisting of all points at which $t_1 \dots t_n$ is invertible.

4.4.8. On the category $\hat{\mathcal{C}}_{\log}$ (resp. $\hat{\mathcal{C}}_{\text{nl}}$), we have a canonical morphism

$$\begin{aligned} \theta : \overline{\mathfrak{M}}_{N,+,\sigma}^{r,n} &\rightarrow \overline{\mathcal{M}}_{N,(+,\sigma)}^{r,n} \quad (\text{resp. } \overline{\mathfrak{M}}_{N,+,\sigma}^{r,n} \rightarrow (\overline{\mathcal{M}}_{N,(+,\sigma)}^{r,n}, W)) \\ (\phi, \iota) &\mapsto ((\psi, \iota'), t_1, \dots, t_n), \end{aligned}$$

where $(\psi, \iota') \in \mathcal{M}_N^r$ is as in 4.4.5 and

$$t_i := \iota(e_{i+r-1})^{-1} \text{ for } 1 \leq i \leq n.$$

Theorem 4.4.9. *Let the notation be as in 4.4.6 and 4.4.7.*

There is a unique open set $\overline{\mathcal{M}}_{N,+,\sigma}^{r,n}$ of $\overline{\mathcal{M}}_{N,(+,\sigma)}^{r,n}$ such that θ induces an isomorphism

$$\overline{\mathfrak{M}}_{N,+,\sigma}^{r,n} \xrightarrow{\cong} \overline{\mathcal{M}}_{N,+,\sigma}^{r,n}$$

of functors on $\hat{\mathcal{C}}_{\log}$ and an isomorphism

$$\overline{\mathfrak{M}}_{N,+,\sigma}^{r,n} \xrightarrow{\cong} (\overline{\mathcal{M}}_{N,+,\sigma}^{r,n}, W)$$

of functors on $\hat{\mathcal{C}}_{\text{nl}}$.

The next 4.4.10 follows from 4.4.9.

Proposition 4.4.10. *Let M be $\overline{\mathfrak{M}}_N^{r,n}$ or $\overline{\mathfrak{M}}_{N,+,\sigma}^{r,n}$. Then the functor M on the category $\hat{\mathcal{C}}_{\text{nl}}$ coincides with the functor $(S, U) \mapsto M(S)$ where the last S is endowed with the log structure associated to U (1.2.4).*

4.5 Iterated Tate uniformizations

We prove Theorem 4.4.9.

We prove Theorem 4.4.9 for $\hat{\mathcal{C}}_{\text{nl}}$ by induction on n (fixing r) by using the method of iterated Tate uniformizations.

4.5.1. For the proof of 4.4.9 for $\hat{\mathcal{C}}_{\text{nl}}$, we may assume $\sigma \in \Sigma_k$. In fact, if $\sigma \subset \tau \in \Sigma_k$ and if an open set $\overline{\mathcal{M}}_{N,+,\tau}^{r,n}$ of $\overline{\mathcal{M}}_{N,(+,\tau)}^{r,n}$ satisfies $\theta : \overline{\mathfrak{M}}_{N,\tau,+}^{r,n} \xrightarrow{\cong} (\overline{\mathcal{M}}_{N,+,\tau}^{r,n}, W)$. then the inverse image $\overline{\mathcal{M}}_{N,+,\sigma}^{r,n}$ of $\overline{\mathcal{M}}_{N,+,\tau}^{r,n}$ under $\overline{\mathcal{M}}_{N,(+,\sigma)}^{r,n} \rightarrow \overline{\mathcal{M}}_{N,(+,\tau)}^{r,n}$ satisfies $\theta : \overline{\mathfrak{M}}_{N,+,\sigma}^{r,n} \xrightarrow{\cong} (\overline{\mathcal{M}}_{N,+,\sigma}^{r,n}, W)$.

Assume $\sigma \in \Sigma_k$.

Lemma 4.5.2. *Let $\sigma \in \Sigma_k = {}_d\Sigma_k$ and let $'\sigma$ is the image of σ under the projection $C_d \rightarrow C_{d-1}$. Then $'\sigma \in {}_{d-1}\Sigma_k$,*

Proof. Assume $n \geq 1$. Let $\tilde{\sigma} \in {}_d\Sigma^{(k)}$ be the cone corresponding to σ . This $\tilde{\sigma}$ is described as the set of $s \in C_d$ satisfying the following conditions for some α (3.1.4): $q^h s_j - s_i \in \alpha(h, i, j)$ ($1 \leq j < i \leq d-1, 0 \leq h \leq k-1$). Here $\alpha(h, i, j) \in \{\mathbb{R}_{\leq 0}, \{0\}, \mathbb{R}_{\geq 0}\}$. Let $'\tilde{\sigma} \in {}_{d-1}\Sigma^{(k)}$ be the cone of all $s \in C_{d-1}$ satisfying the following conditions for the same α : $q^h s_j - s_i \in \alpha(h, i, j)$ ($1 \leq j < i \leq d-2, 0 \leq h \leq k-1$). That is, $'\tilde{\sigma}$ is the image of $\tilde{\sigma}$ under the projection $C_d \rightarrow C_{d-1} : (s_i)_{1 \leq i \leq d-1} \rightarrow (s_i)_{1 \leq i \leq d-2}$. Define $'\sigma \in {}_{d-1}\Sigma_k$ to be the cone corresponding to $'\tilde{\sigma}$. Then $'\sigma$ is the image of σ under the projection $C_d \rightarrow C_{d-1}$. \square

4.5.3. By induction on n , we may assume that $n \geq 1$ and that we have the open set $\overline{\mathcal{M}}_{N,+,\sigma}^{r,n-1}$ of $\overline{\mathcal{M}}_{N,(+,\sigma)}^{r,n-1}$ and an isomorphism $\theta : \overline{\mathfrak{M}}_{N,+,\sigma}^{r,n-1} \xrightarrow{\cong} (\overline{\mathcal{M}}_{N,+,\sigma}^{r,n-1}, 'W)$, where $'W$ is defined in the same way as W .

Let V be the open set of $\overline{\mathcal{M}}_{N,(+,\sigma)}^{r,n}$ consisting of (ψ, t_1, \dots, t_n) such that $(\psi, t_1, \dots, t_{n-1})$ belongs to $\overline{\mathcal{M}}_{N,+,\sigma}^{r,n-1}$. Here we define an open set V' of V . Later we will have $W \subset \overline{\mathcal{M}}_{N,+,\sigma}^{r,n} \subset V' \subset V \subset \overline{\mathcal{M}}_{N,(+,\sigma)}^{r,n}$. We will have a map $\nu : \overline{\mathcal{M}}_{N,+,\sigma}^{r,n} \rightarrow \overline{\mathfrak{M}}_{N,+,\sigma}^{r,n}$ by the method of iterated Tate uniformization, which is almost the inverse of θ .

Let $((\psi, \iota'), t_1, \dots, t_n) \in V$. Then we have $(\iota\phi, \iota') \in \overline{\mathfrak{M}}_{N,+, \sigma}^{r, n-1}$ corresponding to $((\psi, \iota'), t_1, \dots, t_{n-1}) \in \overline{\mathfrak{M}}_{N,+, \sigma}^{r, n-1}$.

For $a \in A/NA$, let $\tilde{a} \in A$ be the unique element such that $\tilde{a} \bmod N = a$ and $|\tilde{a}| < |N|$.

Let $D = \{a = (a_i)_{i=0}^{d-2} \mid a_i \neq 0 \text{ for some } i\}$. Let D_1 be the subset of D consisting of all elements a such that: $|\tilde{a}_i|s_i \leq s_{d-1}$ for all i such that $r \leq i \leq d-2$ and for all $s \in \tilde{\sigma}$. Let $D_2 = D \setminus D_1$. Then if $a \in D_2$, there is i such that $r \leq i \leq d-2$ and such that $|\tilde{a}_i|s_i \geq s_{d-1}$ for all $s \in \tilde{\sigma}$.

If $a \in D_1$, we have an element $g_a := t_n \iota(\sum_{i=0}^{d-2} a_i e_i)$ of $\mathcal{O}_V(V)$. If $a \in D_2$, we have an element $g_a := t_n^{-1} \iota(\sum_{i=0}^{d-2} a_i e_i)^{-1} \in \mathcal{O}_V(V)$. Let V' be the open part of V consisting of points at which $1 - g_a$ are invertible for all $a \in D$. Let $E = \sum_{i=0}^{r-1} (A/NA)e_i$.

Note that $\iota\phi(N)(z) = cz \prod_{a \in E} (z - \iota(a)) \prod_{a \in D \setminus E} (1 - \iota(a)^{-1}z)$ for some $c \in \mathcal{O}_V(V)^\times$. Hence we have $\iota\phi(N)(t_n^{-1}) \neq 0$ on V' . Let

$$t := (\iota\phi(N)(t_n^{-1}))^{-1}.$$

4.5.4. By the definition of V' , if \mathcal{V} is a complete discrete valuation ring with field of fractions K and if we have a morphism $\mathrm{Spf}(\mathcal{V}) \rightarrow V'$ of formal schemes which induces $\mathrm{Spa}(K, \mathcal{V}) \rightarrow W$, then the pullbacks to \mathcal{V} satisfies that the $d-1$ -th component of the image of $(0^{r-1}, v_K(t_1), \dots, v_K(t_n))$ under $\xi_0 \circ \xi_k^{-1} : C_d \rightarrow C_d$ is $v_K(t)$.

4.5.5. We obtain a generalized Drinfeld module ϕ on V' by dividing $\iota\phi$ by the lattice $\iota\phi(AN)(t_n^{-1}) = \iota\phi(A)(t^{-1})$.

For this, we have to check that the assumption in 4.1.8 is satisfied if we take $\iota\phi$ as ψ there (the present t is used as t there). That is, we have to check that $(\iota f)^{-q} t^{(q^{r+n-1}-1)(q-1)}$ belongs to $\mathcal{O}_{V'}$, where ιf is the coefficient of $\iota\phi(T)(z)$ at the highest degree. By 2.6.5, it is sufficient to prove that for \mathcal{V}, K and $\mathrm{Spf}(\mathcal{V}) \rightarrow V'$ as in 4.5.4, we have $qv_K(\iota f) \leq (q^{r+n-1} - 1)(q-1)v_K(t)$.

Let $s \in C_d$ be the element such that $\xi_k(s) = (0^{r-1}, v_K(t_1), \dots, v_K(t_n))$. The formula 3.2.24 for $\iota\phi$ is written as

$$v_K(\iota f) = (q-1) \sum_{i=1}^{n-1} q^{n-1-i} \xi_0(s)_{i+r-1}.$$

By 4.5.4, we have $\xi_0(s)_i \leq \xi_0(s)_{d-1} = v_K(t)$ for all i . Hence $v_K(\iota f) \leq (q-1) \sum_{i=1}^{n-1} q^{n-1-i} v(t) = (q^{n-1} - 1)v_K(t)$. Hence it is sufficient to show $q(q^{n-1} - 1) \leq (q^{r+n-1} - 1)(q-1)$. In fact, $(q^{r+n-1} - 1)(q-1) - q(q^{n-1} - 1) \geq (q^n - 1)(q-1) - q(q^{n-1} - 1) = (q-2)q^n + 1 \geq 0$.

Let $e_{n, n-1}$ be the exponential map for the lattice $\iota\phi(A)(t^{-1})$.

4.5.6. Let \mathcal{V}, K and $\mathrm{Spf}(\mathcal{V}) \rightarrow V'$ be as 4.5.4. Then the pullbacks of $\iota\phi$ and ϕ to \mathcal{V} have the following properties.

Let $\phi \leftrightarrow (\psi, \Lambda)$ and $\iota\phi \leftrightarrow (\psi, \iota\Lambda)$ be the correspondences of Drinfeld over K . Let $e_{n,0}$ be the exponential map for Λ and let $e_{n-1,0}$ be the exponential map for $\iota\Lambda$. We have $e_{n,0} = e_{n, n-1} \circ e_{n-1,0}$ and we have an isomorphism $e_{n-1,0} : \Lambda/\Lambda' \xrightarrow{\cong} \iota\phi(A)(t^{-1})$.

4.5.7. Let the situation be as in 4.5.6. Then by 4.5.6, we have:

Let β be either a non-zero N -torsion point of ϕ or a non-zero N -torsion point of $'\phi$ in K^{sep} (actually all these points belong to K). Then $\beta^{-1}e_{n,n-1}(\beta z) \in \mathcal{V}[[z]]$, and $\beta^{-1}e_{n,n-1}(\beta z) \equiv z \pmod{t_n}$.

4.5.8. By 4.5.7 and by 2.6.5, we have:

For every $a \in D = (\frac{1}{N}A/A)^d \setminus \{0\}$, we have $\iota(a)^{-1}e_{n,n-1}(\iota(a)z) \equiv z \pmod{t_n}$ on V' .

4.5.9. We have the level N structure ι on ϕ by using the level N structure $'\iota$ on $'\phi$ as $\iota(e_i) := e_{n,n-1} \circ '\iota(e_i)$ for $0 \leq i \leq d-2$ and $\iota(e_{d-1}) := e_{n,n-1} \circ '\iota(t_n^{-1})$.

By 4.5.8, we have:

(1) For $1 \leq i \leq n$, $\iota(e_{i+r-1})t_i$ is a unit on V' and $\iota(e_{i+r-1})t_i \equiv 1 \pmod{t_n}$.

4.5.10. Since V' is log regular, the level N structure ι of ϕ satisfies the condition (div) in 4.3.8. We define $\overline{\mathcal{M}}_{N,+,\sigma}^{r,n}$ as the part of V' for which the following (i) holds (this is an open set by (div)).

(i) For $0 \leq i \leq d-1$ and for every $a \in (\frac{1}{N}A/A)^d$ which does not belong to $\sum_{j=1}^{i-1} (A/NA)e_j$, we have $\text{pole}(\iota(a)) \text{pole}(\iota(e_i))^{-1} \in M/\mathcal{O}^\times$ in $M^{\text{gp}}/\mathcal{O}^\times$.

Then the restriction of (ϕ, ι) to $(\overline{\mathcal{M}}_{\Sigma,+,\sigma}^{r,n}, W)$ belongs to $\overline{\mathfrak{M}}_{N,+,\sigma}^{r,n}(\overline{\mathcal{M}}_{N,+,\sigma}^{r,n}, W)$.

4.5.11. The image of $\overline{\mathfrak{M}}_{N,+,\sigma,+}^{r,n} \rightarrow (\overline{\mathcal{M}}_{N,(+,\sigma)}^{r,n}, W)$ is contained in $(\overline{\mathcal{M}}_{N,+,\sigma}^{r,n}, W)$

Proof. This follows from 4.5.8. □

4.5.12. By 4.5.8, the composition $\theta \circ \nu$ is an automorphism of $X := \overline{\mathcal{M}}_{N,+,\sigma}^{r,n}$. It induces the identity morphism of the log formal scheme $(X, \mathcal{O}_X/(t_n))$.

4.5.13. We completes the proof of Theorem 4.4.9 for $\hat{\mathcal{C}}_{\text{nl}}$.

By 4.5.12, it is sufficient to prove the surjectivity of ν .

Let $x \in \overline{\mathfrak{M}}_{N,+,\sigma}^{r,n}$ and let y be the image of x in $(\overline{\mathcal{M}}_{N,+,\sigma}^{r,n}, W)$. Let $z = (\theta \circ \nu)^{-1}(y) \in \overline{\mathfrak{M}}_{N,+,\sigma}^{r,n}$. By 4.5.6 and by 2.6.5, we have $x = \nu(z)$.

4.5.14. The above proof of 4.4.9 for $\hat{\mathcal{C}}_{\text{nl}}$ shows that if (S, U) is an object of $\hat{\mathcal{C}}_{\text{nl}}$ and if an element (ϕ, ι) of $\overline{\mathfrak{M}}_{N,+,\sigma}^{r,n}(S, U)$ is given (σ is as in 4.4.9), then on U , we have local systems of A -modules

$$0 = \Lambda(\phi_0) \subset \Lambda(\phi_1) \subset \cdots \subset \Lambda(\phi_{n-1}) \subset \Lambda(\phi_n) = \Lambda$$

on the étale site of U , where Λ is associated to ϕ by Theorem 4.2.1, ϕ_i for $0 \leq i \leq n$ is the generalized Drinfeld module over S corresponding to $((\psi, \iota'), t_1, \dots, t_i)$, where $((\psi, \iota'), t_1, \dots, t_n)$ corresponds to (ϕ, ι) , and $\Lambda(\phi_i)$ is associated to ϕ_i by 4.2.1. The quotients $\Lambda(\phi_i)/\Lambda(\phi_{i-1})$ ($1 \leq i \leq n$) are constant sheaves and are isomorphic to A .

4.5.15. We prove Theorem 4.4.9 for $\hat{\mathcal{C}}_{\log}$.

Since $\overline{\mathcal{M}}_{N,+,\sigma}^{r,n}$ is log regular, the universal object on $\overline{\mathcal{M}}_{N,+,\sigma}^{r,n}$ gives the morphism $\overline{\mathcal{M}}_{N,+,\sigma}^{r,n} \rightarrow \overline{\mathfrak{M}}_{N,+,\sigma}^{r,n}$ of functors on \mathcal{C}_{\log} . We prove that this is an isomorphism. The surjectivity of this morphism as a morphism of sheaves is straightforward. We prove the injectivity. We have the morphism $\theta : \overline{\mathfrak{M}}_{N,+,\sigma}^{r,n} \rightarrow \overline{\mathcal{M}}_{N,+,\sigma}^{r,n}$ in the converse direction and the composition $\overline{\mathcal{M}}_{N,+,\sigma}^{r,n} \rightarrow \overline{\mathfrak{M}}_{N,+,\sigma}^{r,n} \xrightarrow{\theta} \overline{\mathcal{M}}_{N,+,\sigma}^{r,n}$ is the identity map. This proves the injectivity.

5 Toroidal compactifications

Assume $A = \mathbb{F}_q[T]$. Let $d \geq 1$, $N \in A \setminus \mathbb{F}_q$, and let k be the degree of the polynomial N .

5.1 Satake compactifications

Here we review the Satake compactification $\overline{\mathcal{M}}_{N,\text{Sa}}^d$ of \mathcal{M}_N^d ([18], [25]) following the method of [25]. In [18] and [25], Satake compactification is constructed over F . We simply extend the explicit construction in [25] to that over A or $A[\frac{1}{N}]$.

In the next Section 5.2, we construct our toroidal compactification by using it.

5.1.1. Recall that in the case N has at least two prime divisors (resp. only one prime divisor), we denote by \mathcal{C}_{nl} the category of pairs (S, U) where S is a normal scheme over A (resp. $A[\frac{1}{N}]$) and U is a dense open subspace of S . Recall that

$$\overline{\mathfrak{M}}_{N,\text{Sa}}^d : \mathcal{C}_{\text{nl}} \rightarrow (\text{Sets})$$

is the functor which sends (S, U) to the set of all isomorphism classes of $((\mathcal{L}, \phi), \iota)$ where (\mathcal{L}, ϕ) is a generalized Drinfeld module over S whose restriction $(\mathcal{L}, \phi)|_U$ to U is a Drinfeld module of rank d and ι is a Drinfeld level N structure on $(\mathcal{L}, \phi)|_U$.

We will explain

Proposition 5.1.2. *The pair $(\overline{\mathcal{M}}_{N,\text{Sa}}^d, \mathcal{M}_N^d)$ represents the functor $\overline{\mathfrak{M}}_{N,\text{Sa}}^d$.*

5.1.3. Proposition 5.1.2 follows from the explicit construction of Satake compactification $\overline{\mathcal{M}}_{N,\text{Sa}}^d$ in Pink [25] which we review below.

(1) In the case $N = T$, we have

$$\overline{\mathcal{M}}_{T,\text{Sa}}^d = \text{Proj}(R \otimes_{\mathbb{F}_q} A[\frac{1}{T}]),$$

where R is the \mathbb{F}_q -subalgebra of the rational function field $\mathbb{F}_q(u_0, \dots, u_{d-1})$ generated by f^{-1} for all $f \in V \setminus \{0\}$ where $V := \sum_{i=0}^{d-1} \mathbb{F}_q u_i$. Here the grading of R is such that f^{-1}

for $f \in V \setminus \{0\}$ is of degree 1. Then $\overline{\mathcal{M}}_{T,\text{Sa}}^d$ is normal by [26]. We regard \mathcal{M}_T^d as the open set of $\overline{\mathcal{M}}_{T,\text{Sa}}^d$ consisting of all points at which $f^{-1}g$ are invertible for all $f, g \in V \setminus \{0\}$. The universal object $((\mathcal{L}, \phi), \iota)$ on $(\overline{\mathcal{M}}_{T,\text{Sa}}^d, \mathcal{M}_T^d)$ is defined as follows: \mathcal{L} is $\mathcal{O}(-1)$ of the projective scheme $\overline{\mathcal{M}}_{T,\text{Sa}}^d$. Then $\mathcal{L} \subset \mathcal{O}f$ for all $f \in V$, and locally, $\mathcal{L} = \mathcal{O}f$ for some $f \in V \setminus \{0\}$. The generalized Drinfeld module ϕ is characterized by

$$\phi(T)(z) = Tz \prod_{f \in V \setminus \{0\}} (1 - f^{-1}z),$$

and the level structure ι sends $(T^{-1}a_i \bmod A)_{0 \leq i \leq d-1}$ to $\sum_{i=0}^{d-1} a_i u_i$ for $a_i \in \mathbb{F}_q$. This $(\overline{\mathcal{M}}_{T,\text{Sa}}^d, \mathcal{M}_T^d)$ represents the functor $\overline{\mathfrak{M}}_{T,\text{Sa}}^d$.

- (2) The compactification $\overline{\mathcal{M}}_{N,\text{Sa}}^d \otimes_A A[\frac{1}{N}]$ of $\mathcal{M}_N^d \otimes A[\frac{1}{N}]$ is obtained as follows. Let k be the degree of the polynomial N . Via the ring homomorphism $\mathbb{F}_q[T] \rightarrow \mathbb{F}_q[T]$ over \mathbb{F}_q which sends T to N , a Drinfeld module of rank d for the latter $\mathbb{F}_q[T]$ is regarded as a Drinfeld module of rank dk for the former $\mathbb{F}_q[T]$. This gives a finite morphism $\mathcal{M}_N^d \otimes_A A[\frac{1}{N}] \rightarrow \mathcal{M}_T^{dk}$. Let Z be the image of this morphism, let \bar{Z} be the topological closure of Z in $\overline{\mathcal{M}}_{T,\text{Sa}}^{dk}$, and let $\overline{\mathcal{M}}_{N,\text{Sa}}^d \otimes_A A[\frac{1}{N}]$ be the integral closure of \bar{Z} in $\mathcal{M}_N^d \otimes_A A[\frac{1}{N}]$. The universal object $((\mathcal{L}, \phi), \iota)$ on $\mathcal{M}_N^d \otimes_A A[\frac{1}{N}]$ extends uniquely to the universal object on $\overline{\mathcal{M}}_{N,\text{Sa}}^d \otimes_A A[\frac{1}{N}]$. We see that $(\overline{\mathcal{M}}_{N,\text{Sa}}^d \otimes_A A[\frac{1}{N}], \mathcal{M}_N^d \otimes_A A[\frac{1}{N}])$ with the universal object represents the restriction of the functor $\overline{\mathfrak{M}}_{N,\text{Sa}}^d$ to the full subcategory of \mathcal{C}_{nl} consisting of all objects on which N is invertible. By the construction, $\overline{\mathcal{M}}_{N,\text{Sa}}^d \otimes_A A[\frac{1}{N}]$ is proper over $A[\frac{1}{N}]$. Note that this defines $\overline{\mathcal{M}}_{N,\text{Sa}}^d = \overline{\mathcal{M}}_{N,\text{Sa}}^d \otimes_A A[\frac{1}{N}]$ if N has only one prime divisor, but we have not yet defined $\overline{\mathcal{M}}_{N,\text{Sa}}^d$ if N has at least two prime divisors.
- (3) In the case N has at least two prime divisors, $\overline{\mathcal{M}}_{N,\text{Sa}}^d$ is obtained as follows. Write $N = N_1 N_2$ where $N_1, N_2 \in A \setminus \mathbb{F}_q$, $(N_1, N_2) = 1$. Let $\overline{\mathcal{M}}_{N,\text{Sa}}^d \otimes_A A[\frac{1}{N_i}]$ be the integral closure of $\overline{\mathcal{M}}_{N_i,\text{Sa}}^d \otimes_A A[\frac{1}{N_i}]$ in $\mathcal{M}_N^d \otimes_A A[\frac{1}{N_i}]$. Then it represents the restriction of the functor $\overline{\mathfrak{M}}_{N,\text{Sa}}^d$ to full subcategory of \mathcal{C}_{nl} consisting of objects on which N_i is invertible. Hence $\overline{\mathcal{M}}_{N,\text{Sa}}^d \otimes_A A[\frac{1}{N_i}] \otimes_{A[\frac{1}{N_i}]} A[\frac{1}{N}] = \overline{\mathcal{M}}_{N,\text{Sa}}^d \otimes_A A[\frac{1}{N}]$ for $i = 1, 2$. Hence we get the union $\overline{\mathcal{M}}_{N,\text{Sa}}^d$ over A of the schemes $\overline{\mathcal{M}}_{N,\text{Sa}}^d \otimes_A A[\frac{1}{N_i}]$ over $A[\frac{1}{N_i}]$ ($i = 1, 2$). Then $(\overline{\mathcal{M}}_{N,\text{Sa}}^d, \mathcal{M}_N^d)$ represents $\overline{\mathfrak{M}}_{N,\text{Sa}}^d$. By the construction, $\overline{\mathcal{M}}_{N,\text{Sa}}^d$ is proper over A .

5.2 The special toroidal compactification

Let k be the degree of the polynomial N over \mathbb{F}_q .

In this Section 5.2, we will construct the special toroidal compactification $\overline{\mathcal{M}}_{N,\Sigma_k}^d$ of \mathcal{M}_N^d , and will see that the pair $(\overline{\mathcal{M}}_{N,\Sigma_k}^d, \mathcal{M}_N^d)$ represents the functor $\overline{\mathfrak{M}}_N^d = \overline{\mathfrak{M}}_{N,\Sigma_k}^d$ (3.4.4) on the category

\mathcal{C}_{nl} . We also prove that $\overline{\mathcal{M}}_{N,\Sigma_k}^d$ is log regular and $\overline{\mathcal{M}}_{N,\Sigma_k}^d \otimes_A A[\frac{1}{N}]$ is log smooth over $A[\frac{1}{N}]$. For the proofs of log regularity and log smoothness, we use the formal moduli theory in Section 4.4.

5.2.1. We construct it as follows. Let $S = \overline{\mathcal{M}}_{N,S_a}^d \supset U = \mathcal{M}_N^d$, and let M_S be the associated log structure (1.2.4) on S . Let $((\mathcal{L}, \phi), \iota)$ be the universal generalized Drinfeld module over (S, U) of rank d with level N structure.

For each $a \in (\frac{1}{N}A/A)^d$, let I_a be the ideal of \mathcal{O}_S which is locally generated by $f \in M_S \subset \mathcal{O}_S$ such that the class of f in M_S/\mathcal{O}_S^\times coincide with $\text{pole}(\iota(a))$ (2.6.4). Let the ideal I of \mathcal{O}_S be the product of ideals $I_a + I_b$ for all pairs (a, b) of elements of $(\frac{1}{N}A/A)^d$. Let $\overline{\mathcal{M}}_{N,\Sigma_k}^d$ be the normalization of the blow-up of S along I .

From the properness of $\overline{\mathcal{M}}_{N,S_a}^d$, we have the properness of $\overline{\mathcal{M}}_{N,\Sigma_k}^d$.

By the construction, we have

Proposition 5.2.2. *The pair $(\overline{\mathcal{M}}_{N,\Sigma_k}^d, \mathcal{M}_N^d)$ represents the functor $\overline{\mathfrak{M}}_N^d$ on \mathcal{C}_{nl} .*

Since $\overline{\mathfrak{M}}_N^d = \overline{\mathfrak{M}}_{N,\Sigma_k}^d$, we have

Proposition 5.2.3. *The pair $(\overline{\mathcal{M}}_{N,\Sigma_k}^d, \mathcal{M}_N^d)$ represents the functor $\overline{\mathfrak{M}}_{N,\Sigma_k}^d$ on \mathcal{C}_{nl} .*

5.2.4. For $\sigma \in \Sigma_k$, we have an open set $\overline{\mathcal{M}}_{N,+,\sigma}^d$ of $\overline{\mathcal{M}}_{N,\Sigma_k}^d$ containing \mathcal{M}_N^d such that $(\overline{\mathcal{M}}_{N,+,\sigma}^d, \mathcal{M}_N^d)$ represents the functor $\overline{\mathfrak{M}}_{N,+,\sigma}^d$ on \mathcal{C}_{nl} .

We have an open covering $\overline{\mathcal{M}}_{N,\Sigma_k}^d = \bigcup_{\sigma,g} g(\overline{\mathcal{M}}_{N,+,\sigma}^d)$ where σ ranges over Σ_k and g ranges over $GL_d(A/NA)$.

5.2.5. To study properties of $\overline{\mathcal{M}}_{N,\Sigma_k}^d$ (especially its log regularity) by using the formal moduli theory in Section 4, we first review basic relations of schemes, formal schemes, and adic spaces.

5.2.6. Let S be a locally Noetherian scheme, let Y be a closed subscheme of S , let \hat{S} be formal completion of S along Y , and let $\tilde{S} = \hat{S}_{\text{adic}}$ be the adic space associated to \hat{S} . We have canonical morphisms of locally ringed spaces

$$\pi : (\tilde{S}, \mathcal{O}_{\tilde{S}}) \rightarrow S, \quad \pi^+ : (\tilde{S}, \mathcal{O}_{\tilde{S}}^+) \rightarrow S$$

whose underlying maps $\tilde{S} \rightarrow S$ need not coincide, defined as follows.

Recall that for a locally ringed space X and for a ring R ,

(*) a morphism $X \rightarrow \text{Spec}(R)$ of locally ringed spaces corresponds to a ring homomorphism $R \rightarrow \Gamma(X, \mathcal{O}_X)$ in the one-to-one manner.

From an adic space X , we have two locally ringed spaces (X, \mathcal{O}_X) and (X, \mathcal{O}_X^+) .

In the case S is an affine scheme $\text{Spec}(R)$ and $I \subset R$ is the ideal defining Y , π is the morphism $(\tilde{S}, \mathcal{O}_{\tilde{S}}) \rightarrow S$ corresponding to the ring homomorphism $R \rightarrow \Gamma(\tilde{S}, \mathcal{O}_{\tilde{S}}) = \hat{R} = \varprojlim_n R/I^n$ by (*), and π^+ is the morphism $(\tilde{S}, \mathcal{O}_{\tilde{S}}^+) \rightarrow S$ corresponding to the ring homomorphism $R \rightarrow$

$\Gamma(\tilde{S}, \mathcal{O}_{\tilde{S}}^+) = \hat{R}$ by (*). In this case, the map $\pi : \tilde{S} \rightarrow S$ sends $x \in \tilde{S}$ to the prime ideal $\{f \in R \mid |f(x)| = 0\}$ of R , and the map $\pi^+ : \tilde{S} \rightarrow S$ sends $x \in \tilde{S}$ to the prime ideal $\{f \in R \mid |f(x)| < 1\}$ of R .

The morphism π^+ coincides with the composition $\tilde{S} \rightarrow \hat{S} \rightarrow S$, where the first morphism is the one discussed in 4.3.1 and the second morphism is the evident one.

By the above descriptions of the morphisms π and π^+ in the case S is affine, we see that for a generalized Drinfeld module over S , the generalized Drinfeld module over $\mathcal{O}_{\tilde{S}}$ on \tilde{S} induced via π coincides with that induced via π^+ .

Example 5.2.7. Let $S = \text{Spec}(\mathbb{F}_q[T])$ and $Y = \text{Spec}(\mathbb{F}_q[T]/(T))$. Then $\hat{S} = \text{Spf}(\mathbb{F}_q[[T]])$ and $\hat{S}_{\text{adic}} = \text{Spa}(\mathbb{F}_q[[T]], \mathbb{F}_q[[T]])$ consists of two points s, η such that s is in the closure of η . We have

$$\pi(\eta) = (0), \quad \pi^+(\eta) = \pi(s) = \pi^+(s) = (T).$$

5.2.8. Let $r \geq 1$ and $n \geq 0$ be integers such that $d = r + n$. Let $\sigma \in \Sigma_k$ and let S be the open set of $\overline{\mathcal{M}}_{N,+,\sigma}^d$ consisting of all points at which the fiber of the universal generalized Drinfeld module has rank $\geq r$, and let Y be the closed subset of S consisting of all points at which the fiber of the universal generalized Drinfeld module has rank r . Let \hat{S} be the formal completion of S along Y , and consider the object (\hat{S}, U) of $\hat{\mathcal{C}}_{\text{nl}}$ where U is the inverse image of \mathcal{M}_N^d under $\pi : \tilde{S} := \hat{S}_{\text{adic}} \rightarrow S$ (5.2.6). Let (X, V) be an object of $\hat{\mathcal{C}}_{\text{nl}}$. We have a canonical bijection between the following two sets.

(i) The set of morphisms $X \rightarrow \hat{S}$ of formal schemes over A .

(ii) The set of morphisms $X \rightarrow S$ of locally ringed spaces over A (that is, morphisms $X \rightarrow S$ of formal schemes over A where we regard S as a formal scheme with the sheaf \mathcal{O}_S of discrete rings) whose image is contained in Y .

Hence we have a canonical bijection between the following two sets (i) and (ii).

(i') The set of morphisms $(X, V) \rightarrow (\hat{S}, U)$ in $\hat{\mathcal{C}}_{\text{nl}}$.

(ii') The set of morphisms $X \rightarrow S$ of locally ringed spaces over A whose image is contained in Y such that the composite map $X_{\text{adic}} \rightarrow \tilde{S} \xrightarrow{\pi} S$ induces $V \rightarrow U$.

The set (ii') is identified with $\overline{\mathfrak{M}}_{N,+,\sigma}^{r,n}(X, V)$. Hence (\hat{S}, U) is canonically isomorphic to the object $(\overline{\mathcal{M}}_{N,+,\sigma}^{r,n}, W)$ in Theorem 4.4.9 for $\hat{\mathcal{C}}_{\text{nl}}$ which represents $\overline{\mathfrak{M}}_{N,+,\sigma}^{r,n}$.

Hence by 4.4.9 for $\hat{\mathcal{C}}_{\text{nl}}$, we have that the pair $(\overline{\mathcal{M}}_{N,\Sigma_k}^d, \mathcal{M}_N^d)$ has the property stated in 1.3.6 (3). Hence the pair $(\overline{\mathcal{M}}_{N,\Sigma_k}^d, \mathcal{M}_N^d)$ is log regular and the scheme $\overline{\mathcal{M}}_{N,\Sigma_k}^d \otimes_A A[\frac{1}{N}]$ is smooth over $A[\frac{1}{N}]$.

5.3 General toroidal compactifications

5.3.1. Let $((\mathcal{L}, \phi), \iota)$ be the universal generalized Drinfeld module over $(\overline{\mathcal{M}}_{N,\Sigma_k}^d, \mathcal{M}_N^d)$ of rank d with level N structure. Then for the base $(e_i)_{0 \leq i \leq d-1}$ as in (1) of 1.2.11, the family $(\text{pole}(\iota(e_i)))_{1 \leq i \leq d-1}$,

which is independent of the choice of $(e_i)_{0 \leq i \leq d-1}$, defines an element of $[\Sigma_k](\overline{\mathcal{M}}_{N,\Sigma_k}^d)$. That is, we have a morphism $\overline{\mathcal{M}}_{N,\Sigma_k}^d \rightarrow [\Sigma_k]$ of functors on \mathcal{C}_{\log} . Consider the fs log scheme

$$\overline{\mathcal{M}}_{N,\Sigma}^d := \overline{\mathcal{M}}_{N,\Sigma_k}^d \times_{[\Sigma_k]} [\Sigma * \Sigma_k]$$

(2.5.12). This is proper and log étale over $\overline{\mathcal{M}}_{N,\Sigma_k}^d$. In the case Σ is a subdivision of Σ_k , it is $\overline{\mathcal{M}}_{N,\Sigma_k}^d \times_{[\Sigma_k]} [\Sigma]$.

Since a log étale scheme over a log regular scheme is log regular, we have that $\overline{\mathcal{M}}_{N,\Sigma}^d$ is log regular. From the properness of $\overline{\mathcal{M}}_{N,\Sigma_k}^d$, we have the properness of $\overline{\mathcal{M}}_{N,\Sigma}^d$. From the log smoothness of $\overline{\mathcal{M}}_{N,\Sigma_k}^d \otimes_A A[\frac{1}{N}]$, we have the log smoothness of $\overline{\mathcal{M}}_{N,\Sigma}^d \otimes_A A[\frac{1}{N}]$.

By the construction, the pair $(\overline{\mathcal{M}}_{N,\Sigma}^d, \mathcal{M}_N^d) \in \mathcal{C}_{\text{nl}}$ represents the functor $\overline{\mathfrak{M}}_{N,\Sigma * \Sigma_k}^d = \overline{\mathfrak{M}}_{N,\Sigma}^d$ on \mathcal{C}_{nl} .

5.3.2. For $\sigma \in \Sigma$, we have an open set $\overline{\mathcal{M}}_{N,+,\sigma}^d$ of $\overline{\mathcal{M}}_{N,\Sigma}^d$ containing \mathcal{M}_N^d such that $(\overline{\mathcal{M}}_{N,+,\sigma}^d, \mathcal{M}_N^d)$ represents the functor $\overline{\mathfrak{M}}_{N,+,\sigma}^d$ on \mathcal{C}_{nl} .

We have an open covering $\overline{\mathcal{M}}_{N,\Sigma}^d = \bigcup_{\sigma,g} g(\overline{\mathcal{M}}_{N,+,\sigma}^d)$ where σ ranges over Σ and g ranges over $GL_d(A/NA)$.

By the same arguments as in 5.2.8, we have

Proposition 5.3.3. *Let σ be a finitely generated subcone of C_d which is contained in some cone of Σ_k . Let $r \geq 1$, $n \geq 0$, and assume $d = r + n$. Let S be the open set of $\overline{\mathcal{M}}_{N,+,\sigma}^d$ consisting of points at which the fiber of the universal generalized Drinfeld module (\mathcal{L}, ϕ) is of rank $\geq r$, let Y be the closed subset of S consisting of all points $s \in S$ such that the fiber of (\mathcal{L}, ϕ) at s is of rank r , and let \hat{S} be the formal completion of S along Y . Let U be the inverse image of $\mathcal{M}_N^d \subset \overline{\mathcal{M}}_{N,+,\sigma}^d$ under the map $\pi : \hat{S}_{\text{adic}} \rightarrow \overline{\mathcal{M}}_{N,+,\sigma}^d$ (5.2.6). That is, U is the part of \hat{S}_{adic} on which the log structure is trivial. Then the pair (\hat{S}, U) is isomorphic to the pair $(\overline{\mathcal{M}}_{N,+,\sigma}^{r,n}, W)$ in Theorem 4.4.9 for $\hat{\mathcal{C}}_{\text{nl}}$ which represents the functor $\overline{\mathfrak{M}}_{N,+,\sigma}^{r,n}$ on $\hat{\mathcal{C}}_{\text{nl}}$. In particular (by Theorem 4.4.9 for $\hat{\mathcal{C}}_{\text{nl}}$), there is a canonical morphism $\hat{S} \rightarrow \mathcal{M}_N^r$ and \hat{S} is isomorphic over \mathcal{M}_N^r to an open set of the formal completion of $\mathcal{M}_N^r \times_{\mathbb{F}_p} \text{toric}_{\mathbb{F}_p}(\sigma_n)$ along a closed subscheme with the inverse image of the canonical log structure of $\text{toric}_{\mathbb{F}_p}(\sigma_n)$.*

5.3.4. Theorem 1.3.6 (3) follows from 5.3.3 because \mathcal{M}_N^r is smooth over \mathbb{F}_p , $\mathcal{M}_N^r \otimes_A A[\frac{1}{N}]$ is smooth over $A[\frac{1}{N}]$, and $\text{toric}_{\mathbb{F}_p}(\sigma)$ has the open set $\text{toric}(\sigma') = \mathbf{G}_{m,\mathbb{F}_p}^{r-1} \times_{\mathbb{F}_p} \text{toric}_{\mathbb{F}_p}(\sigma_n)$.

Theorem 5.3.5. *Let $N' \in A \setminus \mathbb{F}_q$ and assume $N'|N$. Let k (resp. k') be the degree of the polynomial N (resp. N'). Let Σ be a finite rational subdivision of Σ_k , and let Σ' be the corresponding finite rational subdivision of $\Sigma_{k',k}$ (3.1.10). Then except the case N has at least two prime divisors and N' has only one prime divisor, $\overline{\mathcal{M}}_{N,\Sigma}^d$ is the integral closure of $\overline{\mathcal{M}}_{N',\Sigma'}^d$ in the function field of \mathcal{M}_N^d . In the exceptional case, the integral closure is $\overline{\mathcal{M}}_{N,\Sigma}^d \otimes_A A[\frac{1}{N'}]$.*

Proof. Assume we are not in the exceptional case. By 3.1.10 and 2.6.5, if X denotes the integral closure of $\overline{\mathcal{M}}_{N',\Sigma'}^d$ in the function field of \mathcal{M}_N^d , (X, \mathcal{M}_N^d) represents the functor $\overline{\mathfrak{M}}_{N,\Sigma}^d$. Hence we have $X = \overline{\mathcal{M}}_{N,\Sigma}^d$. The exceptional case is similar. \square

Proposition 5.3.6. *Let σ be a finitely generated rational subcone of an element of Σ_k . Assume that $\sigma \cap \mathbb{Z}^{d-1}$ is regular (that is, it is generated by m elements as a monoid where m is the dimension of σ). Then the underlying scheme of $\overline{\mathcal{M}}_{N,+,\sigma}^d$ is smooth over A and the complement of \mathcal{M}_N^d in $\overline{\mathcal{M}}_{N,+,\sigma}^d$ is a normal crossing divisor, the underlying scheme of $\overline{\mathcal{M}}_{N,+,\sigma}^d \otimes_A A[\frac{1}{N}]$ is smooth over $A[\frac{1}{N}]$, and the complement of $\mathcal{M}_N^d \otimes_A A[\frac{1}{N}]$ in $\overline{\mathcal{M}}_{N,+,\sigma}^d \otimes_A A[\frac{1}{N}]$ is a relative normal crossing divisor over $A[\frac{1}{N}]$.*

Proof. For a regular σ , $\text{toric}_{\mathbb{F}_p}(\sigma)$ is smooth over \mathbb{F}_p . Hence this follows from Theorem 1.3.6. \square

5.3.7. Since each finite rational cone decomposition of C_d has a finite rational subdivision whose all cones are regular, we obtain Theorem 1.3.5 (3).

Theorem 1.3.7 follows from

Theorem 5.3.8. *Let M be one of $\overline{\mathfrak{M}}_N^d$, $\overline{\mathfrak{M}}_{N,\Sigma}^d$, $\overline{\mathfrak{M}}_{N,+,\sigma}^d$. Then the functor M on the category \mathcal{C}_{nl} coincides with the functor $(S, U) \mapsto M(S)$ where the last S is endowed with the associated log structure (1.2.4).*

Proof. This is because the functor F on \mathcal{C}_{log} is represented by a pair (S, U) such that S is log regular of finite type over A . \square

5.3.9. We prove that the functor $\overline{\mathfrak{M}}_{N,\Sigma}^d: \mathcal{C}_{\text{log}} \rightarrow (\text{Sets})$ is represented by $\overline{\mathcal{M}}_{N,\Sigma}^d$.

The universal log Drinfeld module of rank r with level N structure defines (by taking pullbacks) the morphism $\text{Mor}(S, \overline{\mathcal{M}}_{N,\Sigma}^d) \rightarrow \overline{\mathfrak{M}}_{N,\Sigma}^d(S)$ of functors on \mathcal{C}_{log} .

The functor $\overline{\mathfrak{M}}_{N,\Sigma}^d(S)$ on \mathcal{C}_{log} is a sheaf for the étale topology, and by the definition of log Drinfeld modules of rank d with level N structure, this morphism is a surjection of sheaves on \mathcal{C}_{log} . It remains to prove the injectivity.

Let $a, b: S \rightarrow \overline{\mathcal{M}}_{N,\Sigma}^d$ be morphisms in \mathcal{C}_{log} which induce the same element of $\overline{\mathfrak{M}}_{N,\Sigma}^d(S)$. We prove $a = b$. We may assume that S is of finite type over A . Let $s \in S$, let $\hat{\mathcal{O}}_{S,s}$ be the completion of the local ring $\mathcal{O}_{S,s}$, and consider the morphisms $\hat{a}_s, \hat{b}_s: \text{Spec}(\hat{\mathcal{O}}_{S,s}) \rightarrow \overline{\mathcal{M}}_{N,\Sigma}^d$ of \mathcal{C}_{log} induced by a and b , respectively. Here $\text{Spec}(\hat{\mathcal{O}}_{S,s})$ has the inverse image of the log structure of S . It is sufficient to prove that $\hat{a}_s = \hat{b}_s$. By 5.3.3, these morphisms correspond to morphisms $\hat{a}'_s, \hat{b}'_s: \text{Spf}(\hat{\mathcal{O}}_{S,s}) \rightarrow \overline{\mathcal{M}}_{N,+,\sigma}^{r,n}$ of $\hat{\mathcal{C}}_{\text{log}}$, respectively, for some r, n such that $r + n = d$ and for some $\sigma \in \Sigma$. Here $\text{Spf}(\hat{\mathcal{O}}_{S,s})$ has the inverse image of the log structure of S . Because they give the same element of $\overline{\mathfrak{M}}_{N,+,\sigma}^{r,n}(\text{Spf}(\hat{\mathcal{O}}_{S,s}))$, we have $\hat{a}'_s = \hat{b}'_s$ by Theorem 4.4.9 for $\hat{\mathcal{C}}_{\text{log}}$.

This completes the proof of Theorem 1.3.5.

5.4 Example

Let $d = 3$ and $N = T$.

5.4.1. Let Σ be a finite rational cone decomposition of C_3 . We describe local structures of $\overline{\mathcal{M}}_{T,\Sigma}^3$, irreducible components of the boundary $D := \overline{\mathcal{M}}_{T,\Sigma}^3 \setminus \mathcal{M}_T^3$, and how they intersect.

We have a proper birational morphism $S := \overline{\mathcal{M}}_{T,\Sigma}^3 \rightarrow \mathbf{P}_{A[\frac{1}{T}]}^2$ defined as follows. Let $((\mathcal{L}_\phi, \phi), \iota)$ be the universal log Drinfeld module over S of rank 3 with level T structure. Let $\mathcal{L}' := \sum_{a \in (\frac{1}{T}A/A)^3} \mathcal{O}_S \iota(a) \supset \mathcal{L}_\phi$. Then with the notation $V = \sum_{i=0}^2 \mathbb{F}_q u_i$ in 5.1.3 (1), $\mathcal{L}' = \sum_{f \in V} \mathcal{O}_S f$, \mathcal{L}' is a line bundle, and the surjection $\mathcal{O}_S \otimes_{\mathbb{F}_q} V \rightarrow \mathcal{L}'$ defines $(u_0 : u_1 : u_2) : S \rightarrow \mathbf{P}_{A[\frac{1}{T}]}^2$.

We describe $\overline{\mathcal{M}}_{T,\Sigma}^3$ via this morphism $\overline{\mathcal{M}}_{T,\Sigma}^3 \rightarrow \mathbf{P}_{A[\frac{1}{T}]}^2$.

5.4.2. There are rational numbers α_i with $0 \leq i \leq m$ for some $m \geq 0$ such that

$$1 = \alpha_0 < \alpha_1 < \cdots < \alpha_m$$

and such that Σ consists of the cones

$$\sigma_i := \{(s_1, s_2) \in \mathbb{R}_{\geq 0}^2 \mid \alpha_i s_1 \leq s_2 \leq \alpha_{i+1} s_1\}$$

for $0 \leq i \leq m-1$ and

$$\sigma_m := \{(s_1, s_2) \in \mathbb{R}_{\geq 0}^2 \mid \alpha_m s_1 \leq s_2\},$$

and their faces $\tau_i := \{(s, \alpha_i s) \mid s \in \mathbb{R}_{\geq 0}\}$ for $0 \leq i \leq m$ and $\tau_{m+1} := \{(0, s) \mid s \in \mathbb{R}_{\geq 0}\}$.

For $\sigma \in \Sigma$, let

$$\mathcal{S}(\sigma) := \left\{ \left(\frac{u_0}{u_1} \right)^{b(1)} \left(\frac{u_0}{u_2} \right)^{b(2)} \mid b(i) \in \mathbb{Z}, b(1)s_1 + b(2)s_2 \geq 0 \text{ for all } (s_1, s_2) \in \sigma \right\},$$

and consider the semi-group ring $A[\frac{1}{T}][\mathcal{S}(\sigma)]$. Then $\overline{\mathcal{M}}_{T,+,\sigma}^3$ coincides with the following open subset of $\text{Spec}(A[\frac{1}{T}][\mathcal{S}(\sigma)])$ endowed with the log structure given by $\mathcal{S}(\sigma)$.

In the case σ is one of σ_i ($1 \leq i \leq m-1$), τ_i ($1 \leq i \leq m$), we have

$$\overline{\mathcal{M}}_{T,+,\sigma}^3 = \text{Spec}(A[\frac{1}{T}][\mathcal{S}(\sigma)]).$$

In the case $\sigma = \tau_0$ (resp. τ_{m+1}), we have

$$\overline{\mathcal{M}}_{T,+,\sigma}^3 = \text{Spec}(A[\frac{1}{T}][\mathcal{S}(\sigma)]) \setminus \prod_{v \in \mathbb{F}_q^\times} Y_v,$$

where Y_v is the images of the closed immersion $\text{Spec}(A[\frac{1}{T}]) \rightarrow \text{Spec}(A[\frac{1}{T}][\mathcal{S}(\sigma)])$ given by the ring homomorphism from

$$A[\frac{1}{T}][\mathcal{S}(\sigma)] = A[\frac{1}{T}] \left[\frac{u_0}{u_1}, \left(\frac{u_1}{u_2} \right)^{\pm 1} \right] \quad \left(\text{resp. } A[\frac{1}{T}] \left[\frac{u_1}{u_2}, \left(\frac{u_0}{u_1} \right)^{\pm 1} \right] \right)$$

to $A[\frac{1}{T}]$ over $A[\frac{1}{T}]$ which sends $\frac{u_0}{u_1}$ (resp. $\frac{u_1}{u_2}$) to 0 and $\frac{u_1}{u_2}$ (resp. $\frac{u_0}{u_1}$) to v .

In the case $\Sigma \neq \Sigma_1$ and $\sigma = \sigma_0$ (resp. $\sigma = \sigma_m$), $\overline{\mathcal{M}}_{T,+,\sigma}^3 = \text{Spec}(A[\frac{1}{T}][\mathcal{S}(\sigma)]) \setminus \prod_{v \in \mathbb{F}_q^\times} Y_v$ where Y_v is the above subset of $\text{Spec}(A[\frac{1}{T}][\mathcal{S}(\tau)])$ with $\tau = \tau_0$ (resp. $\tau = \tau_{m+1}$) and we regard $\text{Spec}(A[\frac{1}{T}][\mathcal{S}(\tau)])$ as an open subset of $\text{Spec}(A[\frac{1}{T}][\mathcal{S}(\sigma)])$.

In the case $\Sigma = \Sigma_1$ and $\sigma = \sigma_0$, we have

$$\overline{\mathcal{M}}_{T,+,\sigma}^3 = \text{Spec}(A[\frac{1}{T}][\mathcal{S}(\sigma)]) \setminus (Y \cup Y'),$$

where Y (resp. Y') is the union of Y_v for $v \in \mathbb{F}_q^\times$ in $\text{Spec}(A[\frac{1}{T}][\mathcal{S}(\tau_0)])$ (resp. $\text{Spec}(A[\frac{1}{T}][\mathcal{S}(\tau_1)])$).

5.4.3. The irreducible components of D are

- (i) $D(a)$ for an \mathbb{F}_q -rational point of $\mathbf{P}_{\mathbb{F}_q}^2$.
- (ii) $D(\ell)$ for an \mathbb{F}_q -rational line in $\mathbf{P}_{\mathbb{F}_q}^2$.
- (iii) $D(a, \ell, i)$ for a as in (i) and ℓ as in (ii) such that ℓ contains a , and for an integer i such that $1 \leq i \leq m$.

Here, $D(\ell)$ is the proper transform in $\overline{\mathcal{M}}_{T,\Sigma}^3$ of $\ell \otimes A[\frac{1}{T}] \subset \mathbf{P}_{A[\frac{1}{T}]}^2$. In the case that $\Sigma = \Sigma_1$, the component $D(a)$ is the inverse image of

$$a \otimes A[\frac{1}{T}] \subset \mathbf{P}_{A[\frac{1}{T}]}^2.$$

In general, $D(a)$ is the proper transform in $\overline{\mathcal{M}}_{T,\Sigma}^3$ of $D(a)$ in $\overline{\mathcal{M}}_{T,\Sigma_1}^3$. For $1 \leq i \leq m$, the image of $D(a, \ell, i)$ in $\overline{\mathcal{M}}_{T,\Sigma_1}^3$ is the intersection of $D(a)$ and $D(\ell)$ of $\overline{\mathcal{M}}_{T,\Sigma_1}^3$.

These irreducible components are characterized by the following properties. Let a_0 be the \mathbb{F}_q -rational point $(0 : 0 : 1)$ of $\mathbf{P}_{\mathbb{F}_q}^2$ and let ℓ_0 be the \mathbb{F}_q -rational line $\{(0 : x : y)\}$ in $\mathbf{P}_{\mathbb{F}_q}^2$. We denote $D(a_0)$ by $D(a_0, \ell_0, m+1)$ and denote $D(\ell_0)$ by $D(a_0, \ell_0, 0)$.

- (1) If $g \in \text{GL}_3(\mathbb{F}_q)$ sends a (resp. ℓ) to a' (resp. ℓ'), then g sends $D(a)$ (resp. $D(\ell)$) to $D(a')$ (resp. $D(\ell')$).
- (2) If $g \in \text{GL}_3(\mathbb{F}_q)$ sends (a, ℓ) to (a', ℓ') , then g sends $D(a, \ell, i)$ to $D(a', \ell', i)$.
- (3) For $0 \leq i \leq m+1$, $D(a_0, \ell_0, i)$ is the closure of $\overline{\mathcal{M}}_{T,+,\tau_m}^3 \setminus \mathcal{M}_T^3$.

5.4.4. For $0 \leq i \leq m+1$, $\overline{\mathcal{M}}_{T,+,\tau_i}^3$ is the union of \mathcal{M}_T^3 and the interior of $D(a_0, \ell_0, i)$. (Here for an irreducible component of D , the interior of it means the complement of the intersections with other irreducible components.) For $0 \leq i \leq m$, $\overline{\mathcal{M}}_{T,+,\sigma_i}^3$ is the union of \mathcal{M}_T^3 , the interior of $D(a_0, \ell_0, i)$, the interior of $D(a_0, \ell_0, i+1)$, and the intersection of $D(a_0, \ell_0, i)$ and $D(a_0, \ell_0, i+1)$.

5.4.5. The irreducible components of D intersect as follows:

- $D(a)$ and $D(a')$ do not intersect if $a \neq a'$. $D(\ell)$ and $D(\ell')$ do not intersect if $\ell \neq \ell'$.

- $D(a)$ and $D(\ell)$ if and only if $m = 0$ and ℓ contains a .
- $D(\ell)$ and $D(a, \ell', i)$ intersect if and only if $\ell = \ell'$ and $i = 1$.
- $D(a, \ell, i)$ and $D(a', \ell', i')$ intersect if and only if $a = a'$, $\ell = \ell'$, and $|i - i'| \leq 1$.
- $D(a)$ and $D(a', \ell, i)$ intersect if and only if $a = a'$ and $i = m$.

5.4.6. All irreducible components of D are isomorphic to $\mathbf{P}^1_{A[\frac{1}{T}]}$ and all non-trivial intersections of them are isomorphic to $\text{Spec}(A[\frac{1}{T}])$. These are explained as follows.

- For $0 \leq i \leq m$, write $\alpha_i = c_i d_i^{-1}$ with integers $c_i, d_i > 0$ which are coprime. Let $c_{m+1} = 1$ and $d_{m+1} = 0$. For $0 \leq i \leq m + 1$, let f_i be the invertible element $(\frac{u_0}{u_1})^{c_i} (\frac{u_0}{u_2})^{-d_i}$ of $\mathcal{S}(\tau_i)$.
- For $0 \leq i \leq m + 1$, f_i induces $D(a_0, \ell_0, i) \xrightarrow{\sim} \mathbf{P}^1_{A[\frac{1}{T}]}$.
- For $0 \leq i \leq m$, the intersection of $D(a_0, \ell_0, i)$ and $D(a_0, \ell_0, i + 1)$ is the part of $D(a, \ell, i)$ at which f_i has value ∞ , and it is the part of $D(a, \ell, i + 1)$ at which f_{i+1} has value 0.

5.4.7. For a point s of $\overline{\mathcal{M}}_{T, \Sigma}^3$, $\overline{\mathcal{M}}_{T, \Sigma}^3$ is smooth over $A[\frac{1}{T}]$ at s unless s belongs to the intersection of two irreducible components of D . For a point s of the intersection of $D(a_0, \ell_0, i)$ and $D(a_0, \ell_0, i + 1)$, $\overline{\mathcal{M}}_{T, \Sigma}^3$ is smooth over $A[\frac{1}{T}]$ at s if and only if $c_{i+1}d_i - c_i d_{i+1} = 1$. Hence $\overline{\mathcal{M}}_{T, \Sigma}^3$ is smooth over A if and only if $c_{i+1}d_i - c_i d_{i+1} = 1$ for all $0 \leq i \leq m$.

5.4.8. Concerning the morphism $\overline{\mathcal{M}}_{T, \Sigma_1}^3 \rightarrow \overline{\mathcal{M}}_{T, \text{Sa}}^3$ (5.2.1), the image of $D(\ell)$ is isomorphic to $\text{Spec}(A[\frac{1}{T}])$ for each ℓ , and if we denote the image of $\bigcup_{\ell} D(\ell)$ by Z , then

$$\overline{\mathcal{M}}_{T, \Sigma_1}^3 \setminus \bigcup_{\ell} D(\ell) \rightarrow \overline{\mathcal{M}}_{T, \text{Sa}}^3 \setminus Z$$

is an isomorphism.

Concerning the universal generalized Drinfeld module on $\overline{\mathcal{M}}_{T, \text{Sa}}^3$, the rank of the fiber at $s \in \overline{\mathcal{M}}_{T, \text{Sa}}^3$ is 3 if $s \in \mathcal{M}_T^3$, 1 if $s \in Z$, and 2 otherwise. Hence for the universal generalized Drinfeld module on $\overline{\mathcal{M}}_{T, \Sigma}^3$, which is the pullback of that on $\overline{\mathcal{M}}_{T, \text{Sa}}^3$, the rank of the fiber at $s \in \overline{\mathcal{M}}_{T, \Sigma}^3$ is 3 if $s \in \mathcal{M}_T^3$, 2 if s belongs to the interior of $D(a)$ for some a , and is 1 otherwise.

5.4.9. We describe some open set of the Satake compactification $X := \overline{\mathcal{M}}_{T, \text{Sa}}^3$ explicitly assuming $\mathbb{F}_q = \mathbb{F}_2$ and describe a relation to the toroidal compactification explicitly.

Assume $\mathbb{F}_q = \mathbb{F}_2$.

Take the affine open set $U = \{x \in X \mid u_0 f^{-1} \in \mathcal{O}_{X, x} \text{ for all } f \in V \setminus \{0\}\}$ of X , denote $u_0 u_1^{-1}, u_0 u_2^{-1}, u_0(u_1 + u_2)^{-1} \in \mathcal{O}_U(U)$ by t_1, t_2, t_3 , respectively, and let S be the open set $\{x \in U \mid 1 + t_i \text{ are invertible at } x \text{ for } 1 \leq i \leq 3\}$ of U . We have

$$S = \text{Spec}(A[\frac{1}{T}][t_1, t_2, t_3, (1 + t_1)^{-1}, (1 + t_2)^{-1}, (1 + t_3)^{-1}]/(t_1 t_2 + t_2 t_3 + t_3 t_1)).$$

Then \mathcal{M}_T^3 is identified with the open set of S consisting of points at which $t_1 t_2 t_3$ is invertible. The generalized Drinfeld module $((\mathcal{L}, \phi), \iota)$ over S is understood as $\mathcal{L} = \mathcal{O}_S u_0$,

$$\phi(T)(z u_0) = Tz \left(\prod_{f \in V \setminus \{0\}} (1 - u_0 f^{-1} z) \right) u_0 = Tz(1-z) \left(\prod_{i=1}^3 (1 - t_i z)(1 - t_i(1+t_i)^{-1} z) \right) u_0 \quad (z \in \mathcal{O}_S),$$

$\iota(e_i) = u_i$ for $0 \leq i \leq 2$. The part $t_1 = t_2 = 0$ of S coincides with the set of all points of S at which the fiber of the universal generalized Drinfeld module has rank 1.

For $\Sigma = \Sigma_1$ and $\sigma = \sigma_0 = C_3 = \{s \in \mathbb{R}_{\geq 0}^2 \mid s_1 \leq s_2\}$, $\overline{\mathcal{M}}_{T,+, \sigma}^3$ is identified with the open set of $\text{Spec}(A[\frac{1}{T}][t_1, t_2 t_1^{-1}])$ which is the complement of $Y \cup Y'$ where Y is the closed subset defined by $t_1 = 1 + t_2 t_1^{-1} = 0$ and Y' is the closed subset defined by $1 + t_1 = t_2 t_1^{-1} = 0$, as is explained in 5.4.2. Let M be the open set of $\overline{\mathcal{M}}_{T,+, \sigma}^3$ obtained by inverting $1 + t_2 t_1^{-1}$. The map $\overline{\mathcal{M}}_{T, \Sigma_1}^3 \rightarrow \overline{\mathcal{M}}_{T, S_a}^3$ sends M to S and the pullback of t_3 to M is $t_2(1 + t_2 t_1^{-1})^{-1}$. The divisor $M \setminus \mathcal{M}_T^3$ has two irreducible components, the divisor $t_1 = 0$ and the divisor $t_2 t_1^{-1} = 0$. The former is $D(\ell) \cap M$ for the line $\ell : u_0 = 0$ and the latter is $D(a) \cap M$ for $a = (0 : 0 : 1)$. The former coincides with the set of all points of M at which the fiber of the universal generalized Drinfeld module has rank 1.

The formal scheme $\overline{\mathcal{M}}_{T,+, \sigma}^{1,2}$ is isomorphic over $\overline{\mathcal{M}}_{T, \Sigma_1}^3$ to the t_1 -adic formal completion of M . That is, $\overline{\mathcal{M}}_{T,+, \sigma}^{1,2} = \text{Spf}(H)$ where H is the t_1 -adic completion of $A[\frac{1}{T}][t_1, t_2 t_1^{-1}, (1 + t_2 t_1^{-1})^{-1}]$.

5.4.10. In 5.4.9, let $s \in S$ be a point of the Satake compactification such that $t_i(s) = 0$ for $i = 1, 2, 3$. Then we have neither $t_1 | t_2$ nor $t_2 | t_1$ at s . Hence the universal generalized Drinfeld module over (S, \mathcal{M}_T^3) does not satisfy the condition (div) in 2.7.4.

In the following 5.4.11 and 5.4.12, we consider the local monodromy of the universal generalized Drinfeld module at a point of Satake compactification and at a point of a toroidal compactification, respectively.

5.4.11. In 5.4.9, let s be a point of S such that $t_i(s) = 0$ for $i = 1, 2, 3$. Let R be the completion of the strict henselization of the local ring $\mathcal{O}_{S, s}$ and let Q be the field of fractions of R . Let v be the maximal ideal (T) of A . We consider the action of $\text{Gal}(Q^{\text{sep}}/Q)$ on the three dimensional F_v -vector space $V_v \phi = F_v \otimes_{A_v} T_v \phi$ where $T_v \phi$ is the v -adic Tate module of the universal generalized Drinfeld module ϕ over S viewed as a Drinfeld module over Q . Let ψ be the Drinfeld module over R of rank 1 associated to ϕ (2.3.1). We can regard $T_v \psi \subset T_v \phi$.

Claim. Consider the action of $\text{Gal}(Q^{\text{sep}}/Q)$ on $V_v \phi$ by taking a base $(e_i)_{0 \leq i \leq 2}$ such that e_0 is a base of $V_v \psi$. Then the image of $\text{Gal}(Q^{\text{sep}}/Q)$ in $GL_3(F_v)$ is an open subgroup of

$$\left\{ \begin{pmatrix} 1 & e & f \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, F_v), e, f \in F_v \right\}.$$

Proof of Claim. Let \mathfrak{p}_1 be the prime ideal of R generated by t_2, t_3 and let \mathfrak{p}_2 be the prime ideal of R generated by t_1, t_3 (so $t_i \notin \mathfrak{p}_i$ for $i = 1, 2$). For $i = 1, 2$, let \mathcal{V}_i be the completion of the local ring of R at \mathfrak{p}_i (so \mathcal{V}_i is a complete discrete valuation ring), let K_i be the field of fractions of the strict henselization of \mathcal{V}_i , and fix embeddings $Q^{\text{sep}} \rightarrow K_i^{\text{sep}}$ over Q . Let ϕ_i be the Drinfeld module over \mathcal{V}_i of rank 2 associated to ϕ (2.3.1). We can regard $T_v\psi \subset T_v\phi_i \subset T_v\phi$. Take an A_v -base $(e_i)_{0 \leq i \leq 2}$ of $T_v\phi$ such that e_0 is a base of $T_v\psi$ and the image of e_i in the T -torsion $\phi[T]$ is $u_i = t_i^{-1}u_0$ and such that e_0 and e_i form an A_v -base of $T_v\phi_i$. For $\sigma \in \text{Gal}(K_i^{\text{sep}}/K_i)$, $\sigma - 1$ kills $T_v\phi$, and by the proof of 2.4.2, $(\sigma - 1)A_v e_{3-i}$ for $\sigma \in \text{Gal}(K_i^{\text{sep}}/K_i)$ generate an A_v -submodule of $T_v\phi_i$ of finite index. The action of $\text{Gal}(K_i^{\text{sep}}/K_i)$ on $T_v\phi$ factors through $\text{Gal}(Q^{\text{sep}}/Q)$. The action of $\text{Gal}(Q^{\text{sep}}/Q)$ on $V_v\phi$ is of determinant 1 by the theory of determinants of Drinfeld modules ([1], [11, 2.6.3]). These prove Claim. Here we use the following fact (1).

(1) If U is an open subgroup of $\begin{pmatrix} 1 & F_v \\ 0 & 1 \end{pmatrix}$ and V is an open subgroup of $\begin{pmatrix} 1 & 0 \\ F_v & 1 \end{pmatrix}$, then U and V generate an open subgroup of $\text{SL}_2(F_v)$.

5.4.12. In 5.4.9, let $x \in M = \mathcal{M}_{T,+,\sigma}^3$ with $\sigma = C_3$ be a point at which $t_1(x) = (t_2/t_1)(x) = 0$. Let R be the completion of the strict henselization of the local ring $\mathcal{O}_{M,x}$ and let Q be the field of fractions of R . Let $((\mathcal{L}, \phi), \iota)$ be the universal log Drinfeld module of rank 3 with level T -structure and let $((\psi, \iota'), t_1, t_2)$ be the corresponding section of $\mathcal{M}_{T,+,\sigma}^{1,2}$ over $\text{Spf}(R)$. Let $'\phi$ be the generalized Drinfeld module over R corresponding to $((\psi, \iota'), t_1)$. Let v be as in 5.4.11. We can regard $T_v\psi \subset T_v(' \phi) \subset T_v\phi$.

Claim. Consider the action of $\text{Gal}(Q^{\text{sep}}/Q)$ on the three dimensional F_v -vector space $V_v\phi$ by using a base $(e_i)_{0 \leq i \leq 2}$ such that e_0 is a base of $T_v\psi$ and (e_0, e_1) is a base of $T_v'\phi$. Then the image of $\text{Gal}(Q^{\text{sep}}/Q)$ in $\text{GL}_3(F_v)$ is an open subgroup of

$$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in F_v \right\}.$$

Proof of Claim. Let \mathfrak{p}_1 be the prime ideal of R generated by t_2/t_1 , and let \mathfrak{p}_2 be the prime ideal of R generated by t_1 and $t_2 t_1^{-1}$. Let \mathcal{V}_1 be the valuation ring defined to be the set of all elements of the local ring $R_{\mathfrak{p}_1}$ whose residue classes are in the image of the local ring $R_{\mathfrak{p}_2}$. Let K_1 be the field of fractions of the strict henselization of \mathcal{V}_1 and embed Q^{sep} into K_1^{sep} over Q_1 . We show that the image of $\text{Gal}(K_1^{\text{sep}}/K_1) \subset \text{Gal}(Q^{\text{sep}}/Q)$ in $\text{GL}_3(F_v)$ already has the property in Claim 1. Let \mathfrak{p}'_1 be the prime ideal of \mathcal{V}_1 generated by \mathfrak{p}_1 and let \mathcal{V}_2 be the completion of the local ring of \mathcal{V}_1 at \mathfrak{p}'_1 . Then \mathcal{V}_2 is a complete discrete valuation ring. Let K_2 be the field of fractions of the strict henselization of \mathcal{V}_2 and embed K_1^{sep} into K_2^{sep} over K_1 . Then over \mathcal{V}_2 , $'\phi$ is identified with the Drinfeld module of rank 2 associated to ϕ (2.3.1). The action of $\text{Gal}(Q^{\text{sep}}/Q)$ keeps $V_v\psi$ and $V_v(' \phi)$ and is trivial on $V_v\psi, V_v(' \phi)/V_v\psi$, and on $V_v\phi/V_v(' \phi)$. By the proof of 2.4.2, $(\sigma - 1)A_v e_3$

for $\sigma \in \text{Gal}(K_1^{\text{sep}}/K_1)$ generates a subgroup of $T_v(\phi)$ of finite index, the images of $(\sigma - 1)T_v\phi$ in $T_v(\phi)/T_v\psi$ for $\sigma \in \text{Gal}(K_2^{\text{sep}}/K_2)$ generate a subgroup of finite index, and $(\sigma - 1)T_v(\phi)$ for $\sigma \in \text{Gal}(K_2^{\text{sep}}/K_2)$ generate a subgroup of $T_v\psi$ of finite index. These prove Claim.

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