Iwasawa theory and the Eisenstein ideal

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Abstract

We verify, for each odd prime $p < 1000$, a conjecture of W. McCallum and the author's on the surjectivity of pairings on p-units constructed out of the cup product on the first Galois cohomology group of the maximal unramified outside p extension of $\mathbf{Q}(\mu_p)$ with μ_p-coefficients. In the course of the proof, we relate several Iwasawa-theoretic and Hida-theoretic objects. In particular, we construct a canonical isomorphism between an Eisenstein ideal modulo its square and the second graded piece in an augmentation filtration of a classical Iwasawa module over an abelian pro-p Kummer extension of the cyclotomic $\mathbf{Z}_p$-extension of an abelian field. This Kummer extension arises from the Galois representation on an inverse limit of ordinary parts of first cohomology groups of modular curves that was considered by M. Ohta in order to give another proof of the Iwasawa Main Conjecture in the spirit of that of B. Mazur and A. Wiles. In turn, we relate the Iwasawa module over the Kummer extension to the quotient of the tensor product of the classical cyclotomic Iwasawa module and the Galois group of the Kummer extension by the image of a certain reciprocity map that is constructed out of an inverse limit of cup products up the cyclotomic tower. We give an application to the structure of the Selmer groups of Ohta's modular representation taken modulo the Eisenstein ideal.

1 Introduction

The central object of our study is, for an odd prime p, the cup product in the cohomology of the Galois group $\mathcal{G}$ of the maximal pro-p unramified outside p extension of $F = \mathbf{Q}(\mu_p)$ with μ_p-coefficients,

$$H^1(\mathcal{G}, \mu_p) \otimes H^1(\mathcal{G}, \mu_p) \xrightarrow{\cup} H^2(\mathcal{G}, \mu_p^{\otimes 2}).$$

This cup product is well known to describe the structure of $\mathcal{G}$ modulo the closed normal subgroup generated by pth powers and triple commutators. Moreover, it provides a useful first object in the study of Iwasawa theory over metabelian pro-p unramified outside p extensions of F, as detailed in [Sh2] and Section 4.

Via Kummer theory, the cup product induces a pairing on the p-units $\mathcal{E}_F$ of F,

$$(\,\cdot\,,\,\cdot\,)\colon \mathcal{E}_F \times \mathcal{E}_F \to A_F \otimes \mu_p,$$

where A_F denotes the p-part of the class group of F. In [McS], W. McCallum and the author studied this pairing in detail. For p satisfying Vandiver's Conjecture, which states that A_F equals its (-1)-eigenspace A_F^- under complex conjugation, we conjectured that the image of the pairing $(\,\cdot\,,\,\cdot\,)$ generates $A_F \otimes \mu_p$ [McS, Conjecture 5.3]. Denoting the projection of $(\,\cdot\,,\,\cdot\,)$ to $A_F^- \otimes \mu_p$ by $(\,\cdot\,,\,\cdot\,)^-$, we state the obvious unconditional strengthening of that conjecture.

Conjecture 1.1 (McCallum-Sharifi). *The image of the pairing $(\,\cdot\,,\,\cdot\,)^-$ generates $A_F^- \otimes \mu_p$.*

In fact, the author expects that $(\,\cdot\,,\,\cdot\,)^-$ is itself always surjective. As described in [McS], a positive solution of this conjecture has numerous applications: among others, to R. Greenberg's pseudo-nullity conjecture in Iwasawa theory, to products in K-groups of cyclotomic integer rings, and to Y. Ihara's pro-p Lie algebra arising from the outer representation of Galois on the pro-p fundamental group of the projective line minus three points. In this paper, we focus instead on a relationship between the structure of certain Iwasawa modules over Kummer extensions and the structure of ordinary Hecke algebras of modular forms localized at the Eisenstein ideal. It is this intriguing relationship that enables us to prove the following result.

Theorem 1.2. *The pairing $(\,\cdot\,,\,\cdot\,)$ is surjective for $p < 1000$.*

The cup product pairing has the property that $(x, 1-x) = 0$ if x and $1-x$ are both p-units in F. By imposing relations arising from this fact, McCallum and the author were able to compute a unique nontrivial possibility, up to scalar, for each projection of $(\,\cdot\,,\,\cdot\,)$ to a nontrivial $\Delta = \mathrm{Gal}(F/\mathbf{Q})$-eigenspace of A_F for all $p < 10,000$ (now completed for $p < 25,000$). Therefore, we have a complete computation of the pairing $(\,\cdot\,,\,\cdot\,)$ up to a unit in $(\mathbf{Z}/p\mathbf{Z})[\Delta]$ for all $p < 1000$.

Our method of proof, while given in the context of Iwasawa and Hida theory, is in the spirit of K. Ribet's proof of the converse of Herbrand's theorem [Ri]. We now describe the basic idea at the finite level, first recalling Ribet's work and then explaining our extension of it. Let ω denote the Teichmüller character. If p divides the Bernoulli number B_k for some even k less than p, then there is a congruence between a weight 2, level p, and character ω^{k-2} Eisenstein series and a cuspidal eigenform f of the same weight, level, and character at a prime $\mathfrak{p}$ above p in the ring of coefficients of f. Ribet showed that the fixed field of the kernel of the Galois representation ρ_f attached to f on G_F contains a nontrivial unramified elementary abelian p-extension H of F such that $\mathrm{Gal}(H/F)$ has an ω^{1-k}-action of Δ. In fact, H is fixed by the kernel of the action of G_F on a choice of lattice for ρ_f modulo $\mathfrak{p}$.

For our construction, we assume that p does not divide the Bernoulli number B_{p+1-k} and, for simplicity of this description, that p^2 does not divide B_k. We find a metabelian

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extension M of F in the fixed field of the kernel of ρ_f, the arithmetic of which determines the nontriviality of a certain value of the cup product. Unlike H, the field M cannot be found in the fixed field of the representation on a lattice modulo $\mathfrak{p}$. To find it, we pass to a larger quotient of a lattice by a sublattice intermediate between those determined by $\mathfrak{p}$ and $\mathfrak{p}^2$ (see Section 3). As an extension of F, the field M has maximal abelian subextension equal to the compositum HE of H and a degree p unramified outside p extension $E = F(\eta^{1/p})$ (with $\eta \in \mathcal{E}_F$) of F that has an ω^{k-1}-action of Δ. Over HE, the field M is an unramified elementary abelian p-extension. We show that if no prime above p splits completely in M/HE, then (p, η) does not vanish. To show that no prime above p splits completely in M/HE is equivalent to showing that the pth coefficient of f is not 1 modulo $\mathfrak{p}^2$ (for some f as above), which is computable.

We will consider a rather general set of abelian fields and Dirichlet characters in this article, but let us continue to describe the situation over $K = \mathbf{Q}(\mu_{p^\infty})$ here. Let X_K denote the Galois group of the maximal unramified abelian pro-p extension of K. The Iwasawa Main Conjecture states that the ω^{1-k}-eigenspace of X_K has characteristic ideal generated by a p-adic power series determined by $L_p(\omega^k, s)$. This conjecture was first proven by B. Mazur and A. Wiles in [MW] via a study of the action of the absolute Galois group on "good" quotients of the Jacobians of certain modular curves. The basic idea of their proof is the same as that for Ribet's theorem, but the proof is vastly more involved. A similar but slightly more streamlined proof was given more recently by M. Ohta [O3] using Hida theory (see also [Wi]). Ohta's proof extends methods of M. Kurihara [Ku] and G. Harder and R. Pink [HP], who developed the analogue of Ribet's work in the more technical setting of level one modular forms. It is Ohta's work that we shall call upon in this paper.

We study the action of the absolute Galois group $G_\mathbf{Q}$ on an Eisenstein component $\mathfrak{X}$ of a particular inverse limit of cohomology groups of modular curves previously considered by Ohta [O5] (see Section 2 for the precise definition). Define $\mathfrak{h}$ to be the localization of Hida's ordinary cuspidal Hecke algebra at the maximal ideal containing the Eisenstein ideal $\mathcal{I}$ with character ω^k. Then $\mathfrak{X}$ is an $\mathfrak{h}$-module. If p divides B_k but not B_{p+1-k}, Ohta showed that $\mathfrak{X}$ is free of rank 2 over $\mathfrak{h}$. The fixed field of the kernel of the representation $\rho \colon G_\mathbf{Q} \to \mathrm{Aut}_\mathfrak{h} \mathfrak{X}$ contains a certain abelian pro-p extension L of K that is unramified outside p and totally ramified above p. We can consider X_L, the Galois group of the maximal unramified abelian pro-p extension of L, and its filtration by powers of the augmentation ideal I_G in $\mathbf{Z}_p[[G]]$, where $G = \mathrm{Gal}(L/K)$.

Using the representation ρ, we exhibit a canonical isomorphism between the group of $\mathrm{Gal}(K/\mathbf{Q})$-coinvariants of $I_G X_L / I_G^2 X_L$ and the Eisenstein ideal $\mathcal{I}$ modulo its square (Theorem 3.5). Under this isomorphism, the Frobenius element on the Galois side is identified with $U_p - 1$ on the Hecke side. This yields that the corresponding subquotient of the Galois group Y_L of the maximal unramified abelian pro-p extension of L in which

all primes above p split completely is isomorphic to $\mathcal{I}/\mathcal{I}^2$ modulo the pro-p subgroup generated by $U_p - 1$ (Theorem 3.6). The structure of these latter subquotients can be seen to relate to the ω^{1-k}-eigenspace of the quotient of the classical Iwasawa module X_K by an inverse limit of cup products up the cyclotomic tower. (The precise result, Corollary 5.1, involves a certain reciprocity map constructed out of this inverse limit of cup products in Section 4.) In particular, supposing p does not simultaneously divide B_k and B_{p+1-k} for any even $k < p$, we prove that $(p, \cdot)$ is surjective if and only if $U_p - 1$ generates $\mathcal{I}$ for each k (Theorem 5.6). Thus, to obtain the surjectivity in Theorem 1.2, we verify the latter condition for all $p < 1000$. Note that we do not conjecture that pairing with p is itself always surjective, nor that $U_p - 1$ always generates $\mathcal{I}$.

In Section 6, we give a short discussion of the computation of certain Selmer groups of Hida representations taken modulo the Eisenstein ideal that is made possible by the isomorphisms of Theorems 3.5 and 3.6.

In forthcoming work, we will discuss an as-yet-conjectural relationship between (η_i, η_{k-i}) for special cyclotomic p-units η_i and odd integers i (see [McS, Section 5]), and $L_p(f, \omega^{i-1}, 1)$, which may be viewed as an element of $\mathfrak{p}$, modulo $\mathfrak{p}^2$. Here, f and $\mathfrak{p}$ are as before (and again, for simplicity, $p^2 \nmid B_k$). In this setting, we have $p = \eta_1$, and $L_p(f, 1)$ is explicitly related to $U_p - 1$ by [Ki, Theorem 4.8]. Conjecture 1.1 may be reinterpreted as the statement that the images of these p-adic L-values generate $\mathfrak{p}/\mathfrak{p}^2$ for each k. Theorems 5.2 and 5.6 serve as the first pieces of theoretical evidence for this more powerful conjecture. This conjecture suggests that the cup product pairing is only mildly degenerate, which is supported by the numerical evidence of [McS, Theorem 5.1].

2 Preliminaries

We begin by describing the ordinary Hecke algebras of Hida, which Hida defined and studied in a series of papers (e.g., [H1, H2]). Let $p \geq 5$ be a prime number, and let N be a positive integer prime to p. The ordinary Hecke algebras parameterize ordinary

modular forms of all weights and all levels Np^r with $r \geq 1$. Let us very briefly recall one definition of these Hecke algebras here.

Let $M_k(\Gamma_1(Np^r))$ denote the space of modular forms over $\mathbf{C}$ of level Np^r and weight $k \geq 2$, and let $\mathfrak{H}_k(\Gamma_1(Np^r))$ denote the Hecke subalgebra over $\mathbf{Z}$ of $\mathrm{End}_{\mathbf{C}} M_k(\Gamma_1(Np^r))$. We have natural maps

$$\mathfrak{H}_k(\Gamma_1(Np^{r+1})) \to \mathfrak{H}_k(\Gamma_1(Np^r)).$$

For any commutative ring R with 1, we have a Hecke algebra

$$\mathfrak{H}(N; R) = \varprojlim_{r \geq 1} (\mathfrak{H}_k(\Gamma_1(Np^r)) \otimes_{\mathbf{Z}} R)$$

in the inverse limit, and the Hecke operators in this inverse limit are the inverse limit of the Hecke operators at finite level. We may repeat this construction with $M_k(\Gamma_1(Np^r))$ replaced by $S_k(\Gamma_1(Np^r))$, the space of cusp forms of weight k and level Np^r over $\mathbf{C}$. We obtain Hecke algebras denoted $\mathfrak{h}_k(\Gamma_1(Np^r))$ and $\mathfrak{h}(N; R)$.

Set

$$\mathbf{Z}_{p,N} = \varprojlim_r \mathbf{Z}/p^r N\mathbf{Z}.$$

If $\mathcal{O}$ is a $\mathbf{Z}_p$-algebra, then $\mathfrak{H}(N; \mathcal{O})$ and $\mathfrak{h}(N; \mathcal{O})$ are algebras over $\mathcal{O}[[\mathbf{Z}_{p,N}^{\times}]]$ via the action of the inverse limits of diamond operators. Let $[x] \in \mathcal{O}[[\mathbf{Z}_{p,N}^{\times}]]$ denote the group element associated to $x \in \mathbf{Z}_{p,N}^{\times}$. We use T_l for $l \nmid Np$ and U_l for $l \mid Np$ to denote the usual Hecke operators for prime l in these algebras.

Denote by $H^1(N; \mathcal{O})$ the inverse limit of the cohomology groups $H^1(Y_1(Np^r), \mathcal{O})$, where $Y_1(Np^r)$ is the usual modular curve of level Np^r. The "ordinary parts" of these cohomology groups (i.e., the summands upon which U_p acts invertibly) were studied by Ohta in a series of works [O1]-[O5], and the rest of this section is a partial summary of his results.

Remark. Following Ohta [O2], we consider the action of $\mathfrak{H}(N; \mathcal{O})$ on $H^1(N; \mathcal{O})$ which arises as the inverse limit of the actions of its Hecke operators dual under Weil pairings to the usual actions at each level Np^r. However, we do not give the Hecke operators acting in this manner a separate notation (e.g., $T(l)^*$), since it is the only action on $H^1(N; \mathcal{O})$ we shall use. We also use the original $\mathbf{Z}_{p,N}^{\times}$-action of Hida [H1, p. 552], which is to say that the $\mathbf{Z}_{p,N}^{\times}$-action of Ohta [O1, Section 2.1] on the Hecke algebra $\mathfrak{H}(N; \mathcal{O})$ is a twist of ours by multiplication by χ^{-2}. For any prime l not dividing Np, the operator $T(l, l)$ of Hida satisfies $[l] = l^2 T(l, l)$ in our notation.

From now on, we fix a nontrivial even Dirichlet character $\theta \colon (\mathbf{Z}/Np\mathbf{Z})^{\times} \to \overline{\mathbf{Q}_p}^{\times}$ of conductor N or Np. We let $\mathcal{O}$ be the ring generated over $\mathbf{Z}_p$ by the values of θ. Let $\omega \colon \mathbf{Z}_{p,N}^{\times} \to \mathbf{Z}_p^{\times}$ denote the natural Dirichlet (Teichmüller) character factoring through

$(\mathbf{Z}/p\mathbf{Z})^\times$. Let $\chi\colon \mathbf{Z}_{p,N}^\times \to \mathbf{Z}_p^\times$ denote the natural projection, and let $\kappa = \chi\omega^{-1}$. We will view κ, χ, ω, and θ alternatively as characters $G_\mathbf{Q} \to \mathcal{O}^\times$. For any Dirichlet character ψ, we let $B_{1,\psi}$ denote the associated (first) generalized Bernoulli number.

The following conditions are employed by Ohta in his work.

Hypothesis 1. *We shall make the following assumptions:*

    a. $p \nmid \varphi(N)$, where φ denotes the Euler-phi function,

    b. $\theta|_{(\mathbf{Z}/p\mathbf{Z})^\times} \neq \omega|_{(\mathbf{Z}/p\mathbf{Z})^\times}$,

    c. $\theta|_{(\mathbf{Z}/p\mathbf{Z})^\times} \neq \omega^2|_{(\mathbf{Z}/p\mathbf{Z})^\times}$,

    d. $p \mid B_{1,\theta\omega^{-1}}$,

    e. $p \nmid B_{1,\omega\theta^{-1}}$.

Remark. If $N = 1$, then Hypothesis 1 reduces to parts d and e.

The Eisenstein ideal $\mathfrak{I}_\theta(N; \mathcal{O})$ (resp., $\mathcal{I}_\theta(N; \mathcal{O})$) for the character θ is defined to be the ideal of $\mathfrak{H}(N; \mathcal{O})$ (resp., $\mathfrak{h}(N; \mathcal{O})$) generated by $T_l - 1 - l^{-1}[l]$ and $[l] - \theta(l)[\kappa(l)]$ for $l \nmid Np$, along with $U_l - 1$ for $l \mid Np$. Note that Hypothesis 1a implies that $\mathcal{O}$ is unramified over $\mathbf{Z}_p$. So, let $\mathfrak{M}$ (resp., $\mathfrak{m}$) be the maximal ideal of $\mathfrak{H}(N; \mathcal{O})$ containing $\mathfrak{I}_\theta(N; \mathcal{O})$ (resp., the maximal ideal of $\mathfrak{h}(N; \mathcal{O})$ containing $\mathcal{I}_\theta(N; \mathcal{O})$).

We let $\mathfrak{H} = \mathfrak{H}(N; \mathcal{O})_\mathfrak{M}$, the localization of $\mathfrak{H}(N; \mathcal{O})$ at $\mathfrak{M}$, and similarly, we let $\mathfrak{h} = \mathfrak{h}(N; \mathcal{O})_\mathfrak{m}$. Let $\mathfrak{J}$ (resp., $\mathcal{I}$) denote the image of $\mathfrak{I}_\theta(N; \mathcal{O})$ (resp., $\mathcal{I}_\theta(N; \mathcal{O})$) in $\mathfrak{H}$ (resp., $\mathfrak{h}$). The Hecke algebras $\mathfrak{H}$ and $\mathfrak{h}$ have θ-actions of the diamond operators in $(\mathbf{Z}/Np\mathbf{Z})^\times$. We define $\Lambda_\mathfrak{h}$ to be the $\mathbf{Z}_p$-subalgebra $\mathbf{Z}_p[[1 + p\mathbf{Z}_p]]$ of $\mathfrak{h}$ topologically generated by the (images of the) diamond operators in $\mathfrak{h}$, and we may identify this with the power series ring $\mathbf{Z}_p[[T]]$ by identifying $[1 + p] - 1$ and T. We have an isomorphism of Hecke $\Lambda_\mathfrak{h}$-modules

$$\mathfrak{h}/\mathcal{I} \cong \Lambda_\mathfrak{h}/(g_\theta),$$

where $g_\theta((1 + p)^s - 1) = L_p(\theta, 1 - s)$ for all $s \in \mathbf{Z}_p$ [O4, Corollary A.2.4].

Let $\mathfrak{X} = H^1(N; \mathcal{O}) \otimes_\mathfrak{H} \mathfrak{h}$, which is both a $G_\mathbf{Q}$ and an $\mathfrak{h}$-module. Fix a decomposition group D_p and its inertia subgroup I_p at p inside the absolute Galois group $G_\mathbf{Q}$ of $\mathbf{Q}$. Define $\mathfrak{X}^+$ to be the D_p-module $\mathfrak{X}^{I_p}$. (Propositions 1.2.8 and 1.3.6 of [O3] together imply that $\mathfrak{X}^+$ is D_p-stable.) Pick a cyclic subgroup Δ_p of I_p having order $p-1$ and on which ω is injective. We let $\mathfrak{X}^-$ be the $\mathbf{Z}_p$-submodule of $\mathfrak{X}$ upon which Δ_p acts by $\theta^{-1}\omega$. We have a direct sum decomposition $\mathfrak{X} = \mathfrak{X}^+ \oplus \mathfrak{X}^-$ of $\mathfrak{h}$-modules (as in [O2, (5.3.5)]). Ohta [O5, Theorem 3.3.2] showed that our hypotheses, particularly Hypothesis 1e, imply that $\mathfrak{H}$ is Gorenstein (which also follows from previous work of C. Skinner and A. Wiles [SW]). From this, one obtains that $\mathfrak{X}^+$ and $\mathfrak{X}^-$ are free $\mathfrak{h}$-modules of rank 1 [O5, Section 3.4].

Let

$$\rho\colon G_{\mathbf{Q}} \to \mathrm{Aut}_{\mathfrak{h}}(\mathfrak{X})$$

denote the Galois representation on $\mathfrak{X}$. The fixed field of the kernel of ρ is unramified over $\mathbf{Q}$ outside of the primes dividing Np. We have

$$\det \rho(\sigma) = \theta(\sigma)^{-1}\chi(\sigma)[\kappa(\sigma)^{-1}] \in \mathfrak{h}.$$

We have four maps on $G_{\mathbf{Q}}$ called a, b, c, and d, with

$$a(\sigma) \in \mathrm{End}_{\mathfrak{h}}(\mathfrak{X}^-), \qquad b(\sigma) \in \mathrm{Hom}_{\mathfrak{h}}(\mathfrak{X}^+, \mathfrak{X}^-),$$
$$c(\sigma) \in \mathrm{Hom}_{\mathfrak{h}}(\mathfrak{X}^-, \mathfrak{X}^+), \quad \text{and} \quad d(\sigma) \in \mathrm{End}_{\mathfrak{h}}(\mathfrak{X}^+)$$

for $\sigma \in G_{\mathbf{Q}}$, such that we may represent ρ in matrix form as

$$\rho(\sigma) = \left(\begin{array}{cc} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{array} \right).$$

We remark that for $\sigma \in I_p$, we have

$$\rho(\sigma) = \left(\begin{array}{cc} \det\rho(\sigma) & 0 \\ c(\sigma) & 1 \end{array} \right),$$

and ρ is lower-triangular on an element of a decomposition group D_p containing I_p.

Note that we have canonical isomorphisms $\mathrm{End}_{\mathfrak{h}}(\mathfrak{X}^{\pm}) \cong \mathfrak{h}$ and noncanonical isomorphisms $\mathrm{Hom}_{\mathfrak{h}}(\mathfrak{X}^{\pm}, \mathfrak{X}^{\mp}) \cong \mathfrak{h}$ given by choices of generators of $\mathfrak{X}^+$ and $\mathfrak{X}^-$, which we fix. With these identifications, let B and C denote the ideals of $\mathfrak{h}$ generated by the images of b and c, respectively, under these isomorphisms. Ohta [O5, Section 3.4] used the above-mentioned freeness of $\mathfrak{X}^{\pm}$ to show that $B = \mathcal{I}$ and $C = \mathfrak{h}$ using a method developed by Harder-Pink [HP] and Kurihara [Ku] for level 1 modular forms.

Let F be the fixed field of the kernel of θ on $G_{\mathbf{Q}(\mu_p)}$. Let K denote the cyclotomic $\mathbf{Z}_p$-extension of F. Consider the maps

$$\bar{a}\colon G_{\mathbf{Q}} \to (\mathfrak{h}/\mathcal{I})^{\times} \quad \text{and} \quad \bar{b}\colon G_{\mathbf{Q}} \to B/\mathcal{I}B$$

given by following a and b by the obvious projections. The fixed field of the kernel of

$$\bar{a} = \det \rho \pmod{\mathcal{I}}$$

(see [O3, Lemma 3.3.5]) on $G_{\mathbf{Q}(\mu_p)}$ is K. We have a homomorphism

$$\phi_B\colon G_{\mathbf{Q}} \to B/\mathcal{I}B \rtimes (\mathfrak{h}/\mathcal{I})^{\times},$$

the semi-direct product taken with respect to the natural action, which we can represent in matrix form as

$$\phi_B(\sigma) = \begin{pmatrix} \bar{a}(\sigma) & \bar{b}(\sigma) \\ 0 & 1 \end{pmatrix}$$

for $\sigma \in G_{\mathbf{Q}}$. Let H denote the fixed field of the kernel of ϕ_B on G_F.

Let $\Gamma = \mathrm{Gal}(K/F)$, let $\tilde{\Gamma} = \mathrm{Gal}(K/\mathbf{Q})$, let $\Lambda = \mathcal{O}[[\Gamma]]$, and let $\tilde{\Lambda} = \mathcal{O}[[\tilde{\Gamma}]]$. Let $\tilde{\Delta} = \mathrm{Gal}(F/\mathbf{Q})$, and let $\Delta = \mathrm{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q})$. Note that χ, κ, θ, and ω induce characters on $\tilde{\Gamma}$, Γ, $\tilde{\Delta}$, and Δ, respectively. We make an identification $\Lambda \cong \mathcal{O}[[U]]$ for $U = \gamma - 1$, where $\gamma \in \Gamma$ satisfies $\chi(\gamma) = 1 + p$.

Let A be a $\mathbf{Z}_p[[\tilde{\Gamma}]]$-module. Given a character $\psi \colon \tilde{\Delta} \to \overline{\mathbf{Q}_p}^{\times}$, we let $A^{(\psi)}$ denote the ψ-eigenspace of A. That is, if we let R_ψ denote the ring generated by the values of ψ over $\mathbf{Z}_p$, then

$$A^{(\psi)} = A \otimes_{\mathbf{Z}_p[\tilde{\Delta}]} R_\psi \cong \{a \in A \otimes_{\mathbf{Z}_p} R_\psi \mid \delta a = \psi(\delta)a \text{ for all } \delta \in \tilde{\Delta}\},$$

where the map $\mathbf{Z}_p[\tilde{\Delta}] \to R_\psi$ in the tensor product is the surjection induced by ψ. We remark that $A^{(\psi)}$ has the structure of a $R_\psi[[\tilde{\Gamma}]]$-module. In Appendix A, we discuss the decomposition of $\mathbf{Z}_p[\tilde{\Delta}]$-modules and tensor products of such modules into eigenspaces.

Let X_K denote the Galois group of the maximal unramified abelian pro-p extension of K. Ohta proved the following result on the structure of its $(\omega\theta^{-1})$-eigenspace. Though the proof we mention here uses the Iwasawa main conjecture, Ohta also proved it independently in order to obtain another proof of the Iwasawa main conjecture (for $p \geq 5$).

Theorem 2.1 (Ohta). *The extension H/K is unramified everywhere and $\mathrm{Gal}(H/K) = X_K^{(\omega\theta^{-1})}$. Furthermore, $\bar{b}$ induces a canonical isomorphism*

$$X_K^{(\omega\theta^{-1})} \cong B/\mathcal{I}B$$

of $\tilde{\Lambda}$-modules in which the Galois action on $B/\mathcal{I}B$ is given by $\bar{a}$. In particular, $B/\mathcal{I}B$ has characteristic ideal (g_θ) as a $\Lambda_{\mathfrak{h}}$-module.

Proof. The first two sentences follow from [O3, Theorem 3.3.12]. Let $f_\theta \in \Lambda$ satisfy

$$f_\theta((1+p)^s - 1) = L_p(\theta, s)$$

for $s \in \mathbf{Z}_p$. The Iwasawa main conjecture, as first proven in [MW], states that $X_K^{(\omega\theta^{-1})}$ has characteristic ideal $(f_\theta(U))$ as a Λ-module. Thus, to verify the final sentence, we need merely note that $\bar{a}(\gamma) = (1+p)[1+p]^{-1}$ and $g_\theta((1+p)^s-1) = f_\theta((1+p)^{1-s}-1)$. $\square$

3   The Galois Representation

In this section, we study the maps

$$\bar{c}\colon G_{\mathbf{Q}} \to C/\mathcal{I}C \quad \text{and} \quad \bar{d}\colon G_{\mathbf{Q}} \to (\mathfrak{h}/\mathcal{I}^2)^{\times}$$

arising from c and d by projection. The fixed field of $\bar{c}$ of G_K contains a pro-p abelian extension L of K unramified outside and totally ramified at primes above p, with Galois group G. The image of d is contained in $1+\mathcal{I}$ [O3, Lemma 3.3.5]. We show that $\bar{d}-1$ sets up a canonical isomorphism between $(I_G X_L / I_G^2 X_L)_{\tilde{\Gamma}}$ and $\mathcal{I}/\mathcal{I}^2$, where X_L denotes the Galois group of the maximal unramified abelian pro-p extension of L. If we replace X_L by its maximal quotient Y_L in which all primes above p split completely, then we obtain an isomorphism with the quotient of $\mathcal{I}/\mathcal{I}^2$ by the image of the pro-p group generated by $U_p - 1$.

For any algebraic extension E of $\mathbf{Q}$, we let S_E denote the set of primes above p and any real infinite places, and we let $G_{E,S}$ denote the Galois group of the maximal unramified outside S_E extension of E.

Lemma 3.1. *The Λ-module $X_K^{(\theta\omega^{-1})}$ is trivial, and $X_K^{(\omega\theta^{-1})}$ is free of finite rank over $\mathcal{O}$.*

Proof. Since θ is even, both $X_K^{(\theta\omega^{-1})}$ and $X_K^{(\omega\theta^{-1})}$ have no finite Λ-submodules (see [Wa, Proposition 13.28]). Furthermore, they both have trivial μ-invariant by [FW], and therefore they are both free of finite rank over $\mathcal{O}$. Hypothesis 1e (together with a, b, and c) implies that the p-adic power series associated to $L_p(\omega^2\theta^{-1}, s)$ is a unit (see [Wa, Theorem 5.11]). Iwasawa's Main Conjecture then forces $X_K^{(\theta\omega^{-1})}$ to be finite, hence trivial. □

Now, let us consider the homomorphism $\phi_C\colon G_{\mathbf{Q}} \to C/\mathcal{I}C \rtimes (\mathfrak{h}/\mathcal{I})^{\times}$ that is defined by its matrix representation

$$\phi_C(\sigma) = \begin{pmatrix} \bar{a}(\sigma) & 0 \\ \bar{c}(\sigma) & 1 \end{pmatrix} \tag{3.1}$$

(or, equivalently, by the action of $G_{\mathbf{Q}}$ on $\mathfrak{X}/\mathcal{I}\mathfrak{X}$). Let L denote the fixed field of the kernel of ϕ_C on G_F, and let $G = \mathrm{Gal}(L/K)$. We remark that $G \cong G^{(\theta\omega^{-1})}$, which gives G the structure of a $\tilde{\Lambda}$-module.

For E/K, let Z_E denote the Galois group of the maximal abelian pro-p unramified outside p extension of E. We shall require the following lemma.

Lemma 3.2. *The group $X_K^{(\omega\theta^{-1})}$ has no $\mathcal{O}[[\Gamma']]$-submodule isomorphic to $\mathcal{O}(-1)$ for any nontrivial subgroup Γ' of Γ.*

Proof. Let $K_n = F(\mu_{p^n})$ for $n \geq 1$, and let A_{K_n} denote the p-part of the class group of K_n. We have a canonical exact sequence of $\mathbf{Z}_p[\mathrm{Gal}(K_n/\mathbf{Q})]$-modules:

$$0 \to A_{K_n} \otimes \mu_{p^n} \to H^2(G_{K_n,S}, \mu_{p^n}^{\otimes 2}) \to \bigoplus_{v \in S_{K_n}} \mu_{p^n} \to \mu_{p^n} \to 0. \tag{3.2}$$

Taking inverse limits, we obtain a canonical injection

$$X_K^{(\omega\theta^{-1})}(1) \hookrightarrow \varprojlim H^2(G_{K_n,S}, \mu_{p^n}^{\otimes 2})^{(\omega^2\theta^{-1})} \tag{3.3}$$

(that is an isomorphism under Hypothesis 1b). The group of $\mathrm{Gal}(K/K_n)$-coinvariants of the latter term of (3.3) is isomorphic to the $(\omega^2\theta^{-1})$-eigenspace of $H^2(G_{K_n,S}, \mathbf{Z}_p(2))$ as $G_{F,S}$ has p-cohomological dimension 2. Since $\omega^2\theta^{-1}$ is even, we may replace K_n with its maximal totally real subfield K_n^+, and $H^2(G_{K_n^+,S}, \mathbf{Z}_p(2))$ is finite by a well-known result of C. Soulé [So], as desired. $\qquad\square$

Note that Lemma 3.2 may be stated equivalently as saying that $f_\theta(U)$ is not divisible by any nontrivial divisor of $((U+1)^{p^n} - (1+p)^{-p^n})$ for any n. We are now able to prove the following lemma on the representation ϕ_C.

Lemma 3.3. *The extension L/K is unramified outside p and totally ramified at all primes above p in K. The map $\bar{c}$ induces an isomorphism $G \cong C/\mathcal{I}C$ of $\tilde{\Lambda}$-modules, the action of $\tilde{\Gamma}$ on $C/\mathcal{I}C$ being given by $\bar{a}^{-1}$. We also have $C/\mathcal{I}C \cong \Lambda_\mathfrak{h}/(g_\theta(T))$ as $\Lambda_\mathfrak{h}$-modules.*

Proof. The statement that $G \cong C/\mathcal{I}C$ follows as in [O2, p. 298], the statement on the action is obvious, and the final statement is also obvious as $C = \mathfrak{h}$. Note that we therefore have an isomorphism $G \cong \Lambda/(f_\theta^\circ(U))$ of Λ-modules, where $f_\theta^\circ(U) = f_\theta((U+1)^{-1} - 1)$.

As for the first statement, we note that if v is a prime of K lying above N, the inertia group I_v of v in G cannot be finite and nontrivial, as the Λ-submodule generated by I_v would then be finite (as the set of primes of K above N is finite) and therefore trivial. On the other hand, any ramification at v is tame, so if I_v is infinite, it must be isomorphic to $\mathbf{Z}_p(1)$ as a module over $\mathbf{Z}_p[[\Gamma']]$ for some subgroup $\Gamma' \leq \Gamma$ of finite index. This would imply that $f_\theta^\circ(U)$ shares a nontrivial common divisor with $(U+1)^{p^n} - (1+p)^{p^n}$ for some n, which is not the case by Lemma 3.2. Hence, we see that L/K is unramified outside p. That L/K is totally ramified at primes above p now follows from Lemma 3.1 and the fact that G has a $(\theta\omega^{-1})$-action of $\tilde{\Delta}$. $\qquad\square$

For any algebraic extension E of K, let X_E denote the Galois group of the maximal unramified abelian pro-p extension of E, and let Y_E denote the maximal quotient of X_E

in which all primes above p split completely. Let I_G denote the augmentation ideal in $\mathbf{Z}_p[[G]]$. We have a canonical isomorphism $G \cong I_G/I_G^2$ of $\mathbf{Z}_p[[\tilde{\Gamma}]]$-modules by $\sigma \mapsto \sigma - 1$ (and hence of $\tilde{\Lambda}$-modules, as G has the structure of an $\mathcal{O}$-module).

From now on, we assume the following strengthening of Hypothesis 1b.

Hypothesis 2. *We have that* $\theta^2|_{(\mathbf{Z}/p\mathbf{Z})^\times} \neq \omega^2|_{(\mathbf{Z}/p\mathbf{Z})^\times}$.

Remark. In fact, it would suffice in what follows to assume that θ^2 and ω^2 do not agree on D_p. If $N = 1$, then Hypothesis 2 holds automatically.

Lemma 3.4. *The canonical maps in the following commutative diagram are all $\tilde{\Lambda}$-isomorphisms:*

$$
\begin{array}{ccc}
(X_L/I_G X_L)^{(\omega\theta^{-1})} & \xrightarrow{\;\sim\;} & X_K^{(\omega\theta^{-1})} \\
\downarrow{\wr} & & \downarrow{\wr} \\
(Y_L/I_G Y_L)^{(\omega\theta^{-1})} & \xrightarrow{\;\sim\;} & Y_K^{(\omega\theta^{-1})}.
\end{array}
$$

Proof. For $v \in S_K$, let G_v denote the decomposition group at v in G, let D_v' denote the decomposition group at v in $X_L/I_G X_L$, and let D_v denote the decomposition group at v in X_K. Since, by Lemma 3.3, we know that L/K is unramified outside p and has trivial maximal unramified subextension, we have a commutative diagram with exact rows and columns:

$$
\begin{array}{ccc}
\bigoplus_{v \in S_K} D_v' & \longrightarrow & \bigoplus_{v \in S_K} D_v \\
\downarrow & & \downarrow \\
X_L/I_G X_L & \longrightarrow X_K \longrightarrow & 0 \\
\downarrow & & \downarrow \\
Y_L/I_G Y_L & \longrightarrow Y_K \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

Note that Δ does not permute the primes in S_K. Furthermore, each D_v has a trivial action of Δ, as it is the Galois group of an unramified local extension, and the restriction of $\omega\theta^{-1}$ to Δ is not the identity by Hypothesis 1b. Therefore, the direct sum of the D_v has trivial $(\omega\theta^{-1})$-eigenspace. The same holds for the direct sum of the D_v'.

It now suffices to show that the $(\omega\theta^{-1})$-eigenspace of the kernel $\mathcal{K}$ of $Y_L/I_G Y_L \to Y_K$ is trivial. For this, consider the following commutative diagram with exact columns and

exact rows aside from the first, which is a complex:

$$
\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \downarrow \\
\bigoplus_{v\in S_K} H_2(G_v, \mathbf{Z}_p) & \xrightarrow{\;j\;} & H_2(G, \mathbf{Z}_p) & \longrightarrow & \mathcal{K} \\
\downarrow & & \downarrow & & \downarrow \\
\left(\bigoplus_{w\in S_L} H_1(G_{L_w}, \mathbf{Z}_p)\right)_G & \longrightarrow & (Z_L)_G & \longrightarrow & (Y_L)_G \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{v\in S_K} H_1(G_{K_v}, \mathbf{Z}_p) & \longrightarrow & Z_K & \longrightarrow & Y_K \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{v\in S_K} G_v & \longrightarrow & G & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}
$$

(Note that $H_2(G, \mathbf{Z}_p)$ is isomorphic to the wedge product over $\mathbf{Z}_p$ of two copies of G.) Since $G_v = G$ for each $v \in S_K$, we have that the corestriction map j is surjective. By the Snake Lemma, it remains only to show that the $(\omega\theta^{-1})$-eigenspace of $\bigoplus_{v\in S_K} G_v$ is trivial. For this, we remark as before that its summands are not permuted by Δ. Since $G \cong G^{(\theta\omega^{-1})}$, Hypothesis 2 implies that the $\mathbf{Z}_p[\Delta]$-eigenspace of G_v determined by the restriction of $\omega\theta^{-1}$ is trivial. $\qquad\square$

We now study the homomorphism on $G_{\mathbf{Q}}$ to formal matrices defined as

$$
\phi_D(\sigma) = \left(\begin{array}{cc} \bar{a}(\sigma) & \bar{b}(\sigma) \\ \bar{c}(\sigma) & \bar{d}(\sigma) \end{array}\right)
$$

for $\sigma \in G_{\mathbf{Q}}$. That is, consider the set

$$
\mathcal{N} = \left\{ \left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array}\right) \mid \alpha \in (\mathfrak{h}/\mathcal{I})^\times,\ \beta \in \mathcal{I}/\mathcal{I}^2,\ \gamma \in \mathfrak{h}/\mathcal{I},\ \text{and } \delta \in (\mathfrak{h}/\mathcal{I}^2)^\times \right\},
$$

which we observe forms a group under the usual multiplication of matrices. More specifically, for $\delta \in (\mathfrak{h}/\mathcal{I}^2)^\times$, we let $\bar{\delta}$ denote its image in $(\mathfrak{h}/\mathcal{I})^\times$, and we view $\mathfrak{h}/\mathcal{I}$ and $\mathcal{I}/\mathcal{I}^2$ as $\mathfrak{h}/\mathcal{I}$-modules under multiplication. Our multiplication is:
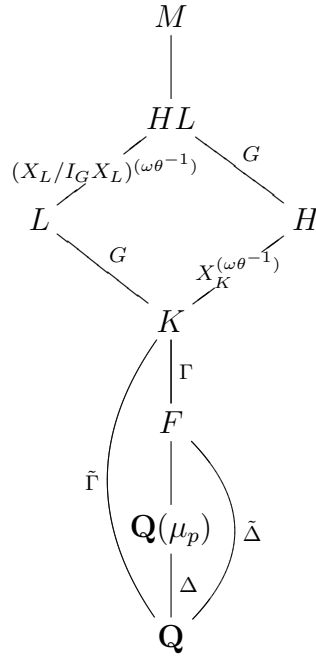
$$
\left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array}\right) \cdot \left(\begin{array}{cc} \alpha' & \beta' \\ \gamma' & \delta' \end{array}\right) = \left(\begin{array}{cc} \alpha\alpha' & \alpha\beta' + \bar{\delta'}\beta \\ \alpha'\gamma + \bar{\delta}\gamma' & \gamma\beta' + \delta\delta' \end{array}\right).
$$

Then ϕ_D is the homomorphism given by following ρ with the projection from the image of ρ in $\mathrm{GL}_2(\mathfrak{h})$ to $\mathcal{N}$, noting again here that $B = \mathcal{I}$.

Let M denote the fixed field of the kernel of ϕ_D on G_F. Although we do not need it, we note that we may use ϕ_D to form a new homomorphism ψ_D, represented in matrix form as

$$\psi_D(\sigma) = \begin{pmatrix} 1 & \bar{c}(\sigma) & \bar{d}(\sigma) - 1 \\ 0 & \bar{a}(\sigma) & \bar{b}(\sigma) \\ 0 & 0 & 1 \end{pmatrix}$$

for $\sigma \in G_{\mathbf{Q}}$, which perhaps gives one a better feeling for the structure of the Galois group $\mathrm{Gal}(M/\mathbf{Q})$. In particular, we may view M as a Heisenberg extension of K. We have a field diagram:

We remark that $G \cong C/\mathcal{I}C$ has an $\bar{a}^{-1}$-action of $\tilde{\Gamma}$, hence a $(\theta\omega^{-1})$-action of $\tilde{\Delta}$, as opposed to the $\bar{a}$-action of $\tilde{\Gamma}$ (resp., $(\omega\theta^{-1})$-action of $\tilde{\Delta}$) on $\mathrm{Gal}(H/K) \cong B/\mathcal{I}B$. Since M/H and M/L are abelian, $\mathrm{Gal}(M/HL)$ too has an action of $\tilde{\Gamma}$, but this action is seen to be trivial from the matrix representation.

Theorem 3.5. *The extension M/HL is unramified everywhere, and*

$$\mathrm{Gal}(M/HL) \cong (I_G X_L / I_G^2 X_L)_{\tilde{\Gamma}}.$$

The map $\bar{d} - 1$ induces a canonical isomorphism of $\mathbf{Z}_p$-modules

$$(I_G X_L / I_G^2 X_L)_{\tilde{\Gamma}} \xrightarrow{\sim} \mathcal{I}/\mathcal{I}^2.$$

Proof. Note the following equality in $\mathcal{I}/\mathcal{I}^2$:

$$\bar{d}(\sigma\tau) - 1 = \bar{c}(\sigma)\bar{b}(\tau) + \bar{d}(\sigma)\bar{d}(\tau) - 1 = \bar{c}(\sigma)\bar{b}(\tau) + (\bar{d}(\sigma) - 1) + (\bar{d}(\tau) - 1)$$

for $\sigma, \tau \in G_{\mathbf{Q}}$. One easily computes that, for $\sigma \in G_H$ and $\tau \in G_L$, we have

$$\bar{d}([\sigma, \tau]) - 1 = \bar{c}(\sigma)\bar{b}(\tau).$$

Since $B = \mathcal{I}$, $C = \mathfrak{h}$, and $\bar{b}$ and $\bar{c}$ are surjective on G_L and G_H, respectively, the group $P = \mathrm{Gal}(M/HL)$ is seen to be equal to the commutator subgroup of $\mathrm{Gal}(M/K)$, and $\bar{d} - 1$ induces a canonical isomorphism

$$P \xrightarrow{\sim} \mathcal{I}/\mathcal{I}^2. \tag{3.4}$$

Now, note that M is an abelian pro-p extension of HL which is unramified outside the primes dividing N. Furthermore, the completion of HL at a prime w above N is the unique unramified $\mathbf{Z}_p$-extension of the completion of F at this prime. If the inertia group P_w of w in P is infinite, then, as the ramification at w is tame, it is isomorphic to $\mathbf{Z}_p(1)$ as a module over $\mathbf{Z}_p[[\Gamma']]$ for some subgroup $\Gamma' \leq \Gamma$ of finite index. However, this is impossible as P has a trivial $\tilde{\Gamma}$-action. Hence, P_w must be finite. However, since P is isomorphic to $X_K^{(\omega\theta^{-1})}$ as a pro-p group by Theorem 2.1 and (3.4), the second statement of Lemma 3.1 forces P_w to be zero. Hence, M/HL is everywhere unramified.

We have now exhibited a surjection

$$(I_G X_L / I_G^2 X_L)_{\tilde{\Gamma}} \twoheadrightarrow P. \tag{3.5}$$

Combining (3.4) and (3.5) and noting Lemma A.3, we have a natural commutative diagram of $\mathbf{Z}_p[[\tilde{\Gamma}]]$-modules:

$$\begin{array}{ccc}
I_G/I_G^2 \otimes_{\mathcal{O}} (X_L/I_G X_L)^{(\omega\theta^{-1})} & \longrightarrow & (I_G X_L/I_G^2 X_L)_{\tilde{\Gamma}} \\
\downarrow{\wr} & & \downarrow \\
C/\mathcal{I}C \otimes_{\mathcal{O}} B/\mathcal{I}B & \longrightarrow & \mathcal{I}/\mathcal{I}^2.
\end{array} \tag{3.6}$$

In fact, the lower horizontal map in (3.6) factors through the tensor product over $\mathfrak{h}$. Of course, since $C = \mathfrak{h}$ and $B = \mathcal{I}$, we have that the map

$$C/\mathcal{I}C \otimes_{\mathfrak{h}} B/\mathcal{I}B \to \mathcal{I}/\mathcal{I}^2 \tag{3.7}$$

is an isomorphism. As the $G_{\mathbf{Q}}$-action on $B/\mathcal{I}B$ is given by $\bar{a}$ and on $C/\mathcal{I}C$ by $\bar{a}^{-1}$, it is easy to see that

$$C/\mathcal{I}C \otimes_{\mathfrak{h}} B/\mathcal{I}B \cong (C/\mathcal{I}C \otimes_{\mathcal{O}} B/\mathcal{I}B)_{\tilde{\Gamma}}.$$

Thus, the right vertical arrow in the diagram (3.6) is also an isomorphism. $\qquad\square$

Theorem 3.6. *Let $\mathcal{V}$ denote the $\mathbf{Z}_p$-span of $U_p - 1$ in $\mathcal{I}$. Then the map $\bar{d} - 1$ induces an isomorphism*

$$(I_G Y_L / I_G^2 Y_L)_{\tilde{\Gamma}} \to \mathcal{I}/(\mathcal{V} + \mathcal{I}^2)$$

of $\mathbf{Z}_p$-modules.

Proof. We know from Theorem 3.5 that $X_L / I_G^2 X_L$ has a $\mathbf{Z}_p[[\mathrm{Gal}(L/\mathbf{Q})]]$-quotient $Z = \mathrm{Gal}(M/L)$ and, therefore, also a quotient Z', which fit into a commutative diagram with exact rows

$$0 \longrightarrow (I_G X_L / I_G^2 X_L)_{\tilde{\Gamma}} \longrightarrow Z \longrightarrow (X_L / I_G X_L)^{(\omega\theta^{-1})} \longrightarrow 0 \qquad (3.8)$$
$$0 \longrightarrow (I_G Y_L / I_G^2 Y_L)_{\tilde{\Gamma}} \longrightarrow Z' \longrightarrow (Y_L / I_G Y_L)^{(\omega\theta^{-1})} \longrightarrow 0.$$

Let $\mathfrak{p}$ be a prime above p in K. We know that L/K is totally ramified at $\mathfrak{p}$ by Lemma 3.3, so there is a unique prime $\mathfrak{P}$ above $\mathfrak{p}$ in L. Since $X_L / I_G^2 X_L$ is abelian, there is a unique decomposition group $D_{\mathfrak{p}}$ at $\mathfrak{P}$ in this Galois group. By Lemma 3.4, the image $\bar{D}_{\mathfrak{p}}$ of $D_{\mathfrak{p}}$ in Z is contained in $(I_G X_L / I_G^2 X_L)_{\tilde{\Gamma}}$. Since $\tilde{\Gamma}$ acts trivially on the latter module, we have that $\bar{D} = \bar{D}_{\mathfrak{p}}$ is the unique decomposition group above p in Z.

We claim that $\bar{D}$ is equal to the kernel of the surjection

$$(I_G X_L / I_G^2 X_L)_{\tilde{\Gamma}} \twoheadrightarrow (I_G Y_L / I_G^2 Y_L)_{\tilde{\Gamma}}.$$

To see this, note that the kernel of the map $X_L / I_G^2 X_L \to Y_L / I_G^2 Y_L$ is the product of the $D_{\mathfrak{p}}$, and hence the kernel of $I_G X_L / I_G^2 X_L \to I_G Y_L / I_G^2 Y_L$ is the intersection of this product with $I_G X_L / I_G^2 X_L$. Now we know that the projection of this product to Z is simply $\bar{D}$, and this is the entire kernel of the map $Z \to Z'$ by (3.8) and the Snake Lemma applied to the diagram

$$0 \longrightarrow I_G(I_G + I_\Gamma) X_L / I_G^2 X_L \longrightarrow I_G X_L / I_G^2 X_L \longrightarrow (I_G X_L / I_G^2 X_L)_{\tilde{\Gamma}} \longrightarrow 0$$
$$0 \longrightarrow I_G(I_G + I_\Gamma) Y_L / I_G^2 Y_L \longrightarrow I_G Y_L / I_G^2 Y_L \longrightarrow (I_G Y_L / I_G^2 Y_L)_{\tilde{\Gamma}} \longrightarrow 0.$$

Hence,

$$(I_G Y_L / I_G^2 Y_L)_{\tilde{\Gamma}} \cong (I_G X_L / I_G^2 X_L)_{\tilde{\Gamma}} / \bar{D}.$$

On the other hand, the map d is a homomorphism on our fixed decomposition group D_p in $G_{\mathbf{Q}}$ which has image U_p on the inverse of a Frobenius at p [O4, Theorem 3.4.2] (using Hypothesis 1c). Since $[F : \mathbf{Q}]$ is prime to p and L/F is totally ramified at primes

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above p, a prime-to-p power of this Frobenius is contained in $D_p \cap G_L$. Also, we have seen that the image of $D_p \cap G_L$ in Z is simply $\bar{D}$. By Theorem 3.5, we obtain immediately that

$$(I_G X_L / I_G^2 X_L)_{\tilde{\Gamma}} / \bar{D} \cong \mathcal{I} / (\mathcal{V} + \mathcal{I}^2),$$

finishing the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

The following example illustrates what occurs in a case in which the class group of F has p-rank 1.

Example. Consider the case of the smallest irregular prime, $p = 37$, which divides the Bernoulli number B_{32}. Let $N = 1$ and $\theta = \omega^{32}$. Here, X_K is its own ω^{-31}-eigenspace, and

$$\mathrm{Gal}(H/K) \cong X_K \cong Y_K \cong \mathbf{Z}_{37}(1 - k)$$

for some $k \equiv 32 \bmod 36$ (inside $\mathbf{Z}_{37} \times \mathbf{Z}/36\mathbf{Z}$). We have $G = \mathrm{Gal}(L/K) \cong \mathbf{Z}_{37}(k - 1)$. Both H and L are generated by cyclotomic 37-units. For instance, the degree 37 subextension of L/K is generated by the 37th root of

$$\eta = \prod_{i=1}^{36} (1 - \zeta_{37}^i)^{i^{30}},$$

where ζ_{37} is a primitive 37th root of 1.

Theorem 3.5 implies

$$G \otimes_{\mathbf{Z}_{37}} X_K \cong I_G / I_G^2 \otimes_{\mathbf{Z}_{37}} X_L / I_G X_L \cong I_G X_L / I_G^2 X_L \cong \mathbf{Z}_{37}$$

as $\tilde{\Lambda}$-modules, and the Galois group $I_G X_L / I_G^2 X_L$ is the commutator subgroup of the Galois group of the extension it defines over K. It turns out that $U_{37} - 1$ generates $\mathcal{I}$, so $I_G X_L / I_G^2 X_L$ is generated by any Frobenius at 37. In other words, $I_G Y_L / I_G^2 Y_L = 0$ and $M = HL$. As we shall see in Theorem 5.6, this also forces the nontriviality of a cup product pairing at the level of F on the 37-units 37 and η.

4   Cup products

In this section, we shall work in a somewhat more abstract setting and construct what we refer to as the S-reciprocity map for an abelian p-power Kummer extension of number fields, which specializes to local reciprocity maps at primes in a fixed set S containing the primes above p and those which ramify. The S-reciprocity map is constructed out of cup products in Galois cohomology with ramification restricted to S and coefficients in

p-power roots of unity. It allows us to give a description of the second graded quotient in the augmentation filtration of the Galois group of the maximal unramified abelian pro-p completely split above p extension of an S-ramified Kummer extension of a cyclotomic $\mathbf{Z}_p$-extension of a number field.

In this section, let p be any prime number, and, for the time being, let F be any number field containing the p^mth roots of unity for some $m \geq 1$. Let S be any finite set of primes of F containing those above p and any real places. For any finite extension E of F, let S_E denote the set of primes of E consisting of the primes above those in S, let $A_{E,S}$ denote the p-part of the S_E-class group of E (and similarly for any extension of F). We let $G_{E,S}$ denote the Galois group of the maximal extension of E unramified outside S_E. Write $\mathcal{O}_E$ and $\mathcal{O}_{E,S}$ for the ring of integers and S_E-integers of E, respectively. Write $\mathrm{Br}_S(E)$ for the S_E-part of the Brauer group of E. In most of the rest of the notation, the set S will be understood. We write $\mathcal{E}_E$ for the group of S_E-units $\mathcal{O}_{E,S}^\times$ in E.

We recall the following exact sequences for $G_{F,S}$-cohomology groups:

$$0 \to \mathcal{E}_F/\mathcal{E}_F^{p^m} \to H^1(G_{F,S}, \mu_{p^m}) \to A_{F,S}[p^m] \to 0 \tag{4.1}$$

and

$$0 \to A_{F,S}/p^m \to H^2(G_{F,S}, \mu_{p^m}) \to \mathrm{Br}_S(F)[p^m] \to 0. \tag{4.2}$$

By Kummer theory and (4.1), the cohomology group $H^1(G_{F,S}, \mu_{p^m})$ is isomorphic to the subgroup $\mathcal{B}_{m,F}$ of $F^\times/F^{\times p^m}$ consisting of reductions of elements $a \in F^\times$ with fractional ideal $a\mathcal{O}_{F,S}$ a p^mth power. We will consider the pairing

$$\langle \, \cdot \, , \, \cdot \, \rangle_{m,F} \colon \mathcal{B}_{m,F} \times \mathcal{B}_{m,F} \to H^2(G_{F,S}, \mu_{p^m}^{\otimes 2})$$

obtained by cup product on $H^1(G_{F,S}, \mu_{p^m})$.

Let Υ be any subgroup of $\mathcal{B}_{m,F}$ that is free over $\mathbf{Z}/p^m\mathbf{Z}$. Let E denote the field defined over F by p^mth roots of lifts of elements of Υ, and let $Q = \mathrm{Gal}(E/F)$. Kummer theory provides a canonical isomorphism

$$Q \xrightarrow{\sim} \mathrm{Hom}(\Upsilon, \mu_{p^m}), \qquad \sigma \mapsto \left(a \mapsto \frac{\sigma(\alpha)}{\alpha}\right), \tag{4.3}$$

where for $a \in \Upsilon$, we use α to denote a p^mth root of a lift of a. Let I_Q denote the augmentation ideal of $\mathbf{Z}_p[Q]$. We have that $Q \cong I_Q/I_Q^2$ in the usual manner, $\sigma \mapsto \sigma - 1$. Tensoring (4.3) with $H^2(G_{F,S}, \mu_{p^m})$ and applying the resulting map to the homomorphism

$$a \mapsto \langle a, b \rangle_{m,F}$$

for $a \in \Upsilon$ and a fixed $b \in \mathcal{E}_F/\mathcal{E}_F^{p^m}$ (for instance), we obtain an element of

$$I_Q/I_Q^2 \otimes H^2(G_{F,S}, \mu_{p^m}).$$

By varying b and lifting to $\mathcal{E}_F$, we have therefore defined a canonical homomorphism

$$\Psi_{m,E/F}\colon \mathcal{E}_F \to I_Q/I_Q^2 \otimes H^2(G_{F,S}, \mu_{p^m}),$$

which one might refer to as the *mth S-reciprocity map* for the extension E/F, as local reciprocity maps can be constructed in an analogous fashion from norm residue symbols. (Note that we need not pass through the subgroup Υ or assume that F contains μ_{p^m} in our construction, but we do so for comparison with the cup product pairing.)

Let $\mathcal{U}_{E/F} = \mathcal{E}_F \cap N_{E/F}E^\times$. We have a p-group

$$\mathcal{P}_{m,E/F} = \Psi_{m,E/F}(\mathcal{U}_{E/F}),$$

and we remark that

$$\mathcal{P}_{m,E/F} \subseteq I_Q/I_Q^2 \otimes \mathcal{A}_{m,E/F},$$

where $\mathcal{A}_{m,E/F}$ is defined to be the image of the norm map $N_{E/F}\colon A_{E,S} \to A_{F,S}/p^m$. (We can take $\mathcal{A}_{m,E/F}$ here instead of $H^2(G_{F,S}, \mu_{p^m})$, or even $A_{F,S}/p^m$, by the fact that global norms are local norms everywhere and [McS, Theorem 2.4]. Note that $(A_{F,S}/p^m)/\mathcal{A}_{m,E/F}$ is canonically isomorphic to a quotient of Q.) The following is immediate from the definition of $\Psi_{m,E/F}$.

Lemma 4.1. *Let $b \in \mathcal{U}_{E/F}$. For $N \geq 0$ and for $\sigma_i \in Q$ and $\mathfrak{a}_i \in \mathcal{A}_{m,E/F}$ with $1 \leq i \leq N$, we have*

$$\Psi_{m,E/F}(b) = \sum_{i=1}^{N}(\sigma_i - 1) \otimes \mathfrak{a}_i$$

if and only if

$$\langle a, b\rangle_{m,F} = \sum_{i=1}^{N} \mathfrak{a}_i \otimes g_{\sigma_i}(a)$$

for all $a \in \Upsilon$, where, for $\sigma \in Q$, we have used g_σ to denote the image of σ under the inverse of the map in (4.3).

Let $\mathcal{D}_{m,E/F}$ be the kernel of $A_{E,S} \to A_{F,S}/p^m$. We define

$$\mathcal{Q}_{m,E/F} = I_Q A_{E,S}/I_Q \mathcal{D}_{m,E/F},$$

which is a quotient of $I_Q A_{E,S}/(p^m + I_Q)I_Q A_{E,S}$.

Theorem 4.2. *There is a canonical isomorphism*

$$(I_Q/I_Q^2 \otimes \mathcal{A}_{m,E/F})/\mathcal{P}_{m,E/F} \xrightarrow{\sim} \mathcal{Q}_{m,E/F}.$$

Proof. We have a natural map

$$f \colon I_Q/I_Q^2 \otimes A_{E,S}/I_Q A_{E,S} \to I_Q A_{E,S}/(p^m + I_Q) I_Q A_{E,S}$$

with

$$f((\sigma - 1) \otimes \bar{\mathfrak{A}}) = (\sigma - 1)\mathfrak{A} \pmod{(p^m + I_Q) I_Q A_{E,S}}.$$

for $\sigma \in Q$ and $\mathfrak{A} \in A_{E,S}$ with image $\bar{\mathfrak{A}}$ in $A_{E,S}/I_Q A_{E,S}$. This descends to a homomorphism

$$h \colon I_Q/I_Q^2 \otimes \mathcal{A}_{m,E/F} \to \mathcal{Q}_{m,E/F}.$$

We must show that h has kernel $\mathcal{P}_{m,E/F}$.

Let $J_{E,S}$ denote the S_E-ideal group of E. Let $b \in \mathcal{U}_{E/F}$, and write $b = N_{E/F}y$ for some $y \in E^\times$. The ideal of $\mathcal{O}_{E,S}$ generated by y may be written in the form

$$y\mathcal{O}_{E,S} = \prod_{i=1}^{t} \mathfrak{A}_i^{1-\sigma_i}$$

with $t \geq 0$, $\mathfrak{A}_i \in J_{E,S}$, and $\sigma_i \in Q$. For $a \in \Upsilon$, let $E_a = F(\alpha)$, where $\alpha^{p^n} = a$. The norm of y to E_a generates

$$\prod_{i=1}^{t} (N_{E/E_a}\mathfrak{A}_i)^{1-\sigma_i|_{E_a}}.$$

Then [McS, Theorem 2.4] implies that

$$\langle a, b \rangle_{m,F} = \sum_{i=1}^{t} N_{E/F}\mathfrak{A}_i \otimes g_{\sigma_i}(a) \tag{4.4}$$

(see also [Sh2, Theorem 4.3]). By Lemma 4.1, we conclude that $h(\Psi_{m,E/F}(b))$ is equal to the image of

$$\sum_{i=1}^{N} (\sigma_i - 1)\mathfrak{A}_i$$

in $\mathcal{Q}_{m,E/F}$, which is 0 by the principality of y. Thus, $\mathcal{P}_{m,E/F}$ is contained in the kernel of h.

On the other hand, if

$$\sum_{i=1}^{N} (\sigma_i - 1)\mathfrak{A}_i$$

has trivial image in $\mathcal{Q}_{m,E/F}$, then there exist $\mathfrak{B}_j \in J_{E,S}$ and $\tau_j \in Q$ for some $1 \leq j \leq M$ and $M \geq 0$ with $N_{E/F}\mathfrak{B}_j$ having trivial class in $A_{F,S}/p^m$ and such that

$$\mathfrak{C} = \prod_{i=1}^{N} \mathfrak{A}_i^{1-\sigma_i} \prod_{j=1}^{M} \mathfrak{B}_j^{1-\tau_j}$$

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is principal. Let $y \in E^\times$ be a generator of $\mathfrak{C}$, and set $b = N_{E/F} y \in \mathcal{E}_F$. As before, (4.4) holds for any $a \in \Upsilon$. Again by Lemma 4.1, we conclude that

$$\Psi_{m,E/F}(b) = \sum_{i=1}^{N} (\sigma_i - 1) \otimes N_{E/F} \mathfrak{A}_i.$$

Hence, the kernel of h equals $\mathcal{P}_{m,E/F}$. $\qquad \square$

We pass to the infinite level. Now let F be any number field and $S = S_F$ a set of primes containing those above p and any real places. Let $K_n = F(\mu_{p^n})$ for all $n \geq 1$, and let $K = F(\mu_{p^\infty})$. Let $\mathcal{B}_{m,K}$ denote the direct limit of the $\mathcal{B}_{m,K_n}$. The pairings $\langle \cdot, \cdot \rangle_{m,K_n}$ for fixed m and increasing n induce, via the usual compatibility of cup products under corestriction, pairings in the limit

$$\mathcal{B}_{m,K} \times \varprojlim_n \mathcal{E}_{K_n} / \mathcal{E}_{K_n}^{p^m} \to \varprojlim_n H^2(G_{K_n,S}, \mu_{p^m}^{\otimes 2}),$$

with the inverse limits being taken with respect to norm and corestriction maps, respectively. Let

$$\mathcal{H}_K = \varprojlim H^2(G_{K_n,S}, \mathbf{Z}_p(1)), \tag{4.5}$$

with the inverse limit taken with respect to corestriction maps. Taking the inverse limit over m, we obtain a pairing

$$\langle \cdot, \cdot \rangle_K \colon \mathcal{B}_K \times \mathcal{U}_K \to \mathcal{H}_K(1),$$

where $\mathcal{B}_K$ denotes the inverse limit of the $\mathcal{B}_{m,K}$ (which contains the p-completion of the p-units in K) and

$$\mathcal{U}_K = \varprojlim (K_n^\times \otimes \mathbf{Z}_p) \cong \varprojlim (\mathcal{E}_{K_n} \otimes \mathbf{Z}_p) \cong \varprojlim_{m,n} \mathcal{E}_{K_n} / \mathcal{E}_{K_n}^{p^m},$$

the inverse limits over n being taken with respect to norm maps. Note that $\mathcal{U}_K$ is simply the group of universal norms of p-units, independent of S, as all primes not over p in K have infinite decomposition groups but trivial inertia groups over F.

Let L be the Kummer extension of K defined by all p-power roots of a pro-p subgroup Υ of $\mathcal{B}_K$, which is defined even though $\mathcal{B}_K$ is only contained in the p-completion of $K^\times$. We remark that $G = \text{Gal}(L/K)$ is necessarily $\mathbf{Z}_p$-torsion free. Since

$$\text{Hom}(\Upsilon, \mathcal{H}_K(1)) \cong G \otimes_{\mathbf{Z}_p} \mathcal{H}_K,$$

we have a natural map

$$\Psi_{L/K} \colon \mathcal{U}_K \to I_G / I_G^2 \otimes_{\mathbf{Z}_p} \mathcal{H}_K \tag{4.6}$$

induced by

$$b \mapsto (a \mapsto \langle a, b \rangle_K)$$

for $b \in \mathcal{U}_K$ and $a \in \Upsilon$. We call $\Psi_{L/K}$ the S-reciprocity map for L/K, and we remark that it satisfies the obvious analogue of Lemma 4.1. For each $n \geq 1$, we may choose a positive integer m_n and an abelian extension L_n/K_n with $(\mathbf{Z}/p^{m_n}\mathbf{Z})$-free Galois group such that $L_n \subset L_{n+1}$ and $L = \cup L_n$. Then one sees easily that

$$\Psi_{L/K}(b) = \varprojlim \Psi_{m_n, L_n/K_n}(b_n)$$

for $b = (b_n) \in \mathcal{U}_K$.

Let

$$\mathcal{U}_{L/K} = \langle b = (b_n) \in \mathcal{U}_K \mid b_n \in (\mathcal{E}_{K_n} \cap N_{L_n/K_n} L_n^\times)/\mathcal{E}_{K_n}^{p^{m_n}} \text{ for all } n \rangle$$

be the group of universal norm sequences in K from the extension L/K (again independent of S containing the primes above p). Recall that Y_L denotes the Galois group of the maximal unramified abelian pro-p extension of K in which all primes above p split completely. Let $Y_{L/K}$ denote the image of the restriction map $Y_L \to Y_K$, and let $\mathcal{D}_{L/K}$ be its kernel. We remark that all primes not above those in S must split in Y_K, so Y_K is in fact the inverse limit of the $A_{K_n, S}$. Furthermore, the image of $b \in \mathcal{U}_{L/K}$ under $\Psi_{L/K}$ lies in $I_G/I_G^2 \otimes_{\mathbf{Z}_p} Y_K$ and is independent of S containing the primes above p and the ramified primes in L/K. Set $\mathcal{P}_{L/K} = \Psi_{L/K}(\mathcal{U}_{L/K})$. We define the following quotient of $I_G Y_L / I_G^2 Y_L$:

$$\mathcal{Q}_{L/K} = I_G Y_L / I_G \mathcal{D}_{L/K}.$$

The following is verified from Theorem 4.2 by taking inverse limits (and by the above remarks, is independent of our choice of S).

Theorem 4.3. *There is a canonical isomorphism*

$$(I_G/I_G^2 \otimes_{\mathbf{Z}_p} Y_{L/K})/\mathcal{P}_{L/K} \xrightarrow{\sim} \mathcal{Q}_{L/K}$$

of $\mathbf{Z}_p$-modules.

We end this section with the following remark.

Remark. Suppose that L is Galois over some subfield F_0 of F, and set $\tilde{\Gamma} = \mathrm{Gal}(K/F_0)$. Then the isomorphism in Theorem 4.3 is an isomorphism of $\mathbf{Z}_p[[\tilde{\Gamma}]]$-modules.

5   Cup products with p

In this section, we combine our results on the Galois representation of Ohta with the cup product construction of the previous section. We return to the setting of Section 3. In

particular, we take $p \geq 5$, we let F be the extension of $\mathbf{Q}(\mu_p)$ cut out by a nontrivial even character θ satisfying Hypotheses 1 and 2, we let K denote the cyclotomic $\mathbf{Z}_p$-extension of F, and we let L be the fixed field of the kernel of $\bar{c}$ on G_K.

We prove in this section that the map given by taking cup products with p at the level of K is surjective if and only if $U_p - 1$ generates the Eisenstein ideal. We describe the computational verification of this generation in the case that $N = 1$ and $p < 1000$. This proves Conjecture 1.1 for these p. Not directly used in this proof but quite striking on its own, we demonstrate in Theorem 5.2 an identification between $U_p - 1$ in the Eisenstein ideal $\mathcal{I}$ modulo its square and the image modulo $\tilde{\Gamma} = \mathrm{Gal}(K/\mathbf{Q})$-coinvariants of a universal norm in K with norm p^{-1} in $\mathbf{Q}$ under the S-reciprocity map $\Psi_{L/K}$ of (4.6). (Here, we have let S_E consist of the primes above p and any real places for any $E/\mathbf{Q}$.)

The following is a corollary of Theorem 4.3.

Corollary 5.1. *There is a canonical isomorphism of $\mathbf{Z}_p[[\tilde{\Gamma}]]$-modules*

$$(I_G/I_G^2 \otimes_{\mathbf{Z}_p} Y_K)^{(1)}/\mathcal{P}_{L/K}^{(1)} \xrightarrow{\sim} (I_G Y_L/I_G^2 Y_L)^{(1)}.$$

Proof. Note that $Y_{L/K} = Y_K$, as L/K is totally ramified at p by Lemma 3.3. Furthermore, by Lemma 3.4, we have $(\mathcal{D}_{L/K}/I_G Y_L)^{(\omega\theta^{-1})} = 0$, which implies that

$$(I_G \mathcal{D}_{L/K}/I_G^2 Y_L)^{(1)} = 0,$$

since $G \cong G^{(\theta\omega^{-1})}$. The result then follows from Theorem 4.3. $\qquad\square$

Let $1 - \zeta = (1 - \zeta_{p^n}) \in \mathcal{U}_K$ for $\zeta = (\zeta_{p^n})$ a generator of the Tate module of μ_{p^∞}. Note that $1 - \zeta$ has norm $p^{[F:\mathbf{Q}(\mu_p)]}$ in $\mathbf{Q}$. Let $\mathfrak{h}$ and $\mathcal{I}$ be as before. We remark that Theorem 3.5 and its proof set up canonical isomorphisms

$$(I_G/I_G^2 \otimes_{\mathbf{Z}_p} Y_K)_{\tilde{\Gamma}} \cong (I_G X_L/I_G^2 X_L)_{\tilde{\Gamma}} \cong \mathcal{I}/\mathcal{I}^2,$$

using in particular (3.6), Lemma A.3, and Lemma 3.4 in obtaining the first isomorphism.

Theorem 5.2. *Under the canonical isomorphism*

$$(I_G/I_G^2 \otimes_{\mathbf{Z}_p} Y_K)_{\tilde{\Gamma}} \cong \mathcal{I}/\mathcal{I}^2$$

the image of $\Psi_{L/K}(1 - \zeta)$ is identified with $[F : \mathbf{Q}(\mu_p)](1 - U_p)$.

Proof. Let $K_n = F(\mu_{p^n})$, and for each $n \geq 1$, let $L_n \subset L$ be an abelian extension of K_n such that $\mathrm{Gal}(L_n/K_n)$ is $(\mathbf{Z}/p^{m_n}\mathbf{Z})$-free for some $m_n \geq 1$, with $L_n \subset L_{n+1}$ and $L = \cup L_n$. Let $b = (b_n) \in \mathcal{U}_{L/K}$, and write

$$b_n = N_{L_n/K_n} y_n \pmod{\mathcal{E}_{K_n}^{p^{m_n}}}$$

for some $y_n \in L_n^\times$ for each $n \geq 1$. Let J_{L_n} denote the ideal group of L_n (which contains $J_{L_n,S}$ as a direct summand), and write

$$y_n \mathcal{O}_{L_n} = \mathfrak{B}_n \prod_{i=1}^{t_n} \mathfrak{A}_{i,n}^{1-\sigma_{i,n}} \bmod p^{m_n} J_{L_n}$$

for some $t_n \geq 0$, $\mathfrak{A}_{i,n} \in J_{L_n,S}$ and $\sigma_{i,n} \in \mathrm{Gal}(L_n/K_n)$ with $1 \leq i \leq t_n$, and $\mathfrak{B}_n$ in the subgroup of J_{L_n} generated by the primes in S_{L_n}. We then have (by [McS, Theorem 2.4] and Lemma 4.1)

$$\Psi_{m_n,L_n/K_n}(b_n) = \sum_{i=1}^{t_n} (\sigma_{i,n} - 1) \otimes N_{L_n/K_n} \mathfrak{A}_{i,n}.$$

Let $\Theta_{L/K}(b)$ denote the image of $\Psi_{L/K}(b)$ in $(I_G X_L / I_G^2 X_L)_{\tilde{\Gamma}}$. As the image of $\Psi_{L/K}(b)$ in $(I_G Y_L / I_G^2 Y_L)_{\tilde{\Gamma}}$ is trivial by Corollary 5.1, we have that $\Theta_{L/K}(b) = v[\mathfrak{P}]$, where $[\mathfrak{P}]$ denotes the image of any chosen prime $\mathfrak{P}$ of L above p in $(I_G X_L / I_G^2 X_L)_{\tilde{\Gamma}}$ and $v \in \mathbf{Z}_p$ denotes the sum of the valuations of b at primes above p in K. In other words, $\Theta_{L/K}(b)$ is the image in this group of the $-rv$th power of a geometric Frobenius Φ_p in D_p, where r is the residue degree of p in $F/\mathbf{Q}$.

Since $(\mathcal{P}_{L/K})_{\tilde{\Gamma}}$ surjects onto the span of $U_p - 1$ in $\mathcal{I}/\mathcal{I}^2$ by Theorem 3.6 and Corollary 5.1, there exists $b \in \mathcal{U}_{L/K}$ with $\bar{d}(\Theta_{L/K}(b)) - 1 = U_p - 1$. Choosing such a b, we have $-rv = 1$, as $\bar{d}(\Phi_p) = U_p$ (again by [O4, Theorem 3.4.2]). Let ϵ denote the natural projection $\mathcal{U}_K \to \mathcal{U}_K^{(1)}$. We remark that $\mathcal{U}_K^{(1)}$ is free of rank 1 over $\mathbf{Z}_p[[\Gamma]]$, generated by $\epsilon(1 - \zeta)$. Since $v = -r^{-1}$ has trivial valuation at p, the element $\epsilon(b)$ also generates $\mathcal{U}_K^{(1)}$ as a $\mathbf{Z}_p[[\Gamma]]$-module. In particular, we have $\mathcal{U}_K^{(1)} = \mathcal{U}_{L/K}^{(1)}$. (It is also possible to see this using cohomological methods.) Since the sum of the valuations of $1 - \zeta$ at primes above p in K is equal to the number g of such primes, we have

$$\Theta_{L/K}(1 - \zeta) = \Theta_{L/K}(\epsilon(1 - \zeta)) = [\mathfrak{P}]^g = \Theta_{L/K}(b)^{-rg} = \Theta_{L/K}(b)^{-[F:\mathbf{Q}(\mu_p)]}.$$

The result follows. $\qquad\qquad\square$

We require a few lemmas. First, we employ Poitou-Tate duality to verify the following reflection principle. Set $\mathcal{U}_F = \mathcal{E}_F \otimes \mathbf{Z}_p$.

Lemma 5.3. *We have $A_{F,S}^{(\omega^2\theta^{-1})} = 0$, and $\mathcal{U}_F^{(\omega^2\theta^{-1})}$ is free of rank 1 over $\mathcal{O}$.*

Proof. By Lemma 3.1, we have that $A_{F,S}^{(\theta\omega^{-1})} = 0$ (or see [L, Theorem 1.3.1]). On the other hand, the $(\theta\omega^{-1})$-eigenspace of $\mathrm{Br}_S(F)[p]$ is trivial by Hypothesis 1b. Hence, (4.2) implies that the $(\theta\omega^{-1})$-eigenspace of $H^2(G_{F,S}, \mu_p)$ is trivial. Since

$$H^i(G_{F,S}, \mu_p)^{(\theta\omega^{-1})} \cong H^i(G_{\mathbf{Q},S}, (\mathcal{O}/p\mathcal{O})(\omega^2\theta^{-1}))$$

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for any $i \geq 0$, the fact that the Euler characteristic of the latter cohomology groups is one implies that the $(\theta\omega^{-1})$-eigenspace of $H^1(G_{F,S}, \mu_p)$ is trivial as well. By Poitou-Tate duality, this implies that the kernel of

$$H^2(G_{F,S}, \mu_p)^{(\omega^2\theta^{-1})} \to \mathrm{Br}_S(F)[p]^{(\omega^2\theta^{-1})}$$

is trivial. As this kernel is the $(\omega^2\theta^{-1})$-eigenspace of $A_{F,S}/p$, we have the first statement.

Now, consider the $(\omega^2\theta^{-1})$-eigenspace of $\bigoplus_{v \in S_F} H^1(G_{F_v}, \mu_p)$. In fact, it follows from a result of C. Greither (see [Sh1, Corollary 2.2]) that under Hypotheses 1b and c, this eigenspace is a one-dimensional vector space over $\mathcal{O}/p\mathcal{O}$. Again applying Poitou-Tate duality, the trivialities demonstrated above and (4.1) imply that

$$(\mathcal{E}_F/\mathcal{E}_F^p)^{(\omega^2\theta^{-1})} \cong H^1(G_{F,S}, \mu_p)^{(\omega^2\theta^{-1})} \cong \Big(\bigoplus_{v \in S_F} H^1(G_{F_v}, \mu_p)\Big)^{(\omega^2\theta^{-1})},$$

finishing the proof. $\qquad\square$

We also have the following.

Lemma 5.4. *Every element of* $\mathcal{U}_F^{(\omega^2\theta^{-1})}$ *is a universal norm from* K.

Proof. Let $\mathcal{U}_{K/F}$ denote the subgroup of $\mathcal{U}_F$ of elements which are universal norms from K. By [Sh3, Corollary A.2], we have an exact sequence

$$0 \to Y_K^\Gamma \otimes_{\mathbf{Z}_p} \Gamma \to \mathcal{U}_F/\mathcal{U}_{K/F} \to \ker\Big(\bigoplus_{v \in S_K} \Gamma_v \to \Gamma\Big) \to (Y_K)_\Gamma \to A_{F,S} \to 0 \qquad (5.1)$$

(which in this case results from [Si, Theorem 6.5]). By Lemma 5.3, we have that $A_{F,S}^{(\omega^2\theta^{-1})} = 0$. Next, note that

$$\Big(\bigoplus_{v \in S_K} \Gamma_v\Big)^{(\omega^2\theta^{-1})} = 0$$

by Hypothesis 1c. Then, (5.1) implies that $(Y_K)_\Gamma^{(\omega^2\theta^{-1})} = 0$, so $Y_K^{(\omega^2\theta^{-1})} = 0$ by Nakayama's Lemma. Applying (5.1) one more time, we see that $\mathcal{U}_F^{(\omega^2\theta^{-1})} = \mathcal{U}_{K/F}^{(\omega^2\theta^{-1})}$, as desired. $\qquad\square$

For $q \geq 2$ even, let $\mathfrak{h}_q(N; \mathcal{O})$ denote the ordinary cuspidal Hecke algebra over $\mathcal{O}$ of level p, weight q, and character $\theta\omega^{-q}$. Let $\mathcal{I}_q(N; \mathcal{O})$ denote the Eisenstein ideal in $\mathfrak{h}_q(N; \mathcal{O})$, which is generated by $U_p - 1$ and $T_l - 1 - \theta(l)\kappa(l)^q l^{-1}$ for primes $l \neq p$. Let $\mathfrak{h}_q$ denote the localization of $\mathfrak{h}_q(N; \mathcal{O})$ at the unique maximal ideal containing $\mathcal{I}_q(N; \mathcal{O})$, and let $\mathcal{I}_q$ denote the resulting Eisenstein ideal in $\mathfrak{h}_q$. Let

$$P_q = [1 + p] - (1 + p)^q \in \Lambda_{\mathfrak{h}}.$$

Then $\mathfrak{h}_q \cong \mathfrak{h}/P_q\mathfrak{h}$ [H2, Corollary 3.2].

Lemma 5.5. *We have $\mathcal{I}_q \cong \mathcal{I}/P_q\mathcal{I}$.*

Proof. We have $\mathcal{I}_q \cong \mathcal{I}/(P_q\mathfrak{h} \cap \mathcal{I})$, and we claim that $P_q\mathfrak{h} \cap \mathcal{I} = P_q\mathcal{I}$. To see this, it suffices to show that P_q does not divide the characteristic ideal of the principal $\Lambda_\mathfrak{h}$-module $\mathfrak{h}/\mathcal{I}$, which we know to be generated by a power series g_θ with $g_\theta((1+p)^s-1) = L_p(\theta, 1-s)$ for $s \in \mathbf{Z}_p$. Since q is a positive even integer, the argument of Lemma 3.2 goes through (with $\mathcal{O}(-1)$ replaced by $\mathcal{O}(1-q)$). $\square$

By Lemma 3.4, we may identify the $(\omega\theta^{-1})$-eigenspaces of X_K and Y_K, as well as of A_F and $A_{F,S}$. Similarly, the $(\omega\theta^{-1})$-eigenspace of $\mathcal{H}_K$, with $\mathcal{H}_K$ as in (4.5), is simply $Y_K^{(\omega\theta^{-1})}$ by Hypothesis 1b. We are now ready to prove the following theorem on the surjectivity of pairing with p under the cup product.

Theorem 5.6. *The following are equivalent:*

   a. $\langle p, \mathcal{U}_K \rangle_K^{(\omega^2\theta^{-1})} = X_K^{(\omega\theta^{-1})}(1)$,

   b. $\langle p, \mathcal{E}_F/\mathcal{E}_F^p \rangle_{1,F}^{(\omega^2\theta^{-1})} = A_F^{(\omega\theta^{-1})} \otimes \mu_p$,

   c. $U_p - 1$ *generates the ideal $\mathcal{I}$ of $\mathfrak{h}$,*

   d. $U_p - 1$ *generates the ideal $\mathcal{I}_q$ of $\mathfrak{h}_q$.*

Proof. We first consider the equivalence of conditions a and b. It follows from Lemma 5.4 that the image of $\langle p, \mathcal{U}_K \rangle_K^{(\omega^2\theta^{-1})}$ under the natural quotient map

$$X_K^{(\omega\theta^{-1})}(1) \to A_F^{(\omega\theta^{-1})} \otimes \mu_p$$

equals $\langle p, \mathcal{E}_F/\mathcal{E}_F^p \rangle_{1,F}^{(\omega^2\theta^{-1})}$ (using p also to denote its image in $\mathcal{E}_F/\mathcal{E}_F^p$). Since this map is given by taking the quotient modulo the action of the maximal ideal of Λ (again using Hypothesis 1c as in Lemma 5.4), Nakayama's Lemma implies that parts a and b are equivalent.

Let $\mathfrak{m}$ denote the maximal ideal of $\mathfrak{h}$ containing $\mathcal{I}$. By Lemma 5.5 and Nakayama's Lemma, conditions c and d are both equivalent to the image of $U_p - 1$ generating $\mathcal{I}/\mathfrak{m}\mathcal{I}$ as an $\mathcal{O}$-module. We prove that this latter condition is equivalent to part b in a series of steps.

Let Q be the quotient of G modulo the Λ-submodule determined by the maximal ideal of Λ. Let E be the subextension of L/F with $Q = \mathrm{Gal}(E/F)$. It follows from the principality of G as a Λ-module that Q has dimension 1 over $\mathcal{O}/p\mathcal{O}$. By Lemma

5.3, E is defined by $(\mathcal{E}_F/\mathcal{E}_F^p)^{(\omega^2\theta^{-1})}$ as a Kummer extension. Let η be a generator of $(\mathcal{E}_F/\mathcal{E}_F^p)^{(\omega^2\theta^{-1})}$ as an $\mathcal{O}$-module. Then

$$\langle p, \mathcal{E}_F/\mathcal{E}_F^p \rangle_{1,F}^{(\omega^2\theta^{-1})} = \langle \eta, \mathcal{E}_F/\mathcal{E}_F^p \rangle_{1,F}^{(\omega^2\theta^{-1})},$$

and both groups are necessarily generated by $\langle p, \eta \rangle_{1,F}$ as an $\mathcal{O}$-module. That is, condition b is equivalent to the statement that $A_F^{(\omega\theta^{-1})} \otimes \mu_p$ is generated by $\langle p, \eta \rangle_{1,F}$ as an $\mathcal{O}$-module.

Lemma A.3 and the isomorphism (3.7) in the proof of Theorem 3.5 yield isomorphisms

$$(I_Q/I_Q^2 \otimes A_F/p)^{(1)} \cong I_Q/I_Q^2 \otimes_{\mathcal{O}} A_F^{(\omega\theta^{-1})}/p \cong C/\mathfrak{m}C \otimes_{\mathcal{O}} B/\mathfrak{m}B \cong \mathcal{I}/\mathfrak{m}\mathcal{I} \qquad (5.2)$$

in the quotient. Since I_Q/I_Q^2 is a one-dimensional $\mathcal{O}/p\mathcal{O}$-vector space, this sets up a noncanonical isomorphism of $\mathcal{O}$-modules

$$A_F^{(\omega\theta^{-1})} \otimes \mu_p \cong \mathcal{I}/\mathfrak{m}\mathcal{I}. \qquad (5.3)$$

If $\mathcal{I}/\mathfrak{m}\mathcal{I}$ has $\mathcal{O}/p\mathcal{O}$-dimension greater than 1, then the image of $U_p - 1$ cannot generate $\mathcal{I}/\mathfrak{m}\mathcal{I}$, nor can $\langle p, \eta \rangle_{1,F}$ generate $A_F^{(\omega\theta^{-1})} \otimes \mu_p$. We may therefore assume that both $\mathcal{O}/p\mathcal{O}$-vector spaces in (5.3) are one-dimensional. We are reduced to proving that $\langle p, \eta \rangle_{1,F}$ is nonzero if and only if the image of $U_p - 1$ in $\mathcal{I}/\mathfrak{m}\mathcal{I}$ is nonzero.

By Theorems 4.3 and 3.6, we have that

$$(I_G/I_G^2 \otimes_{\mathbf{Z}_p} Y_K)_{\tilde{\Gamma}}/(\mathcal{P}_{L/K})_{\tilde{\Gamma}} \cong (I_G Y_L/I_G^2 Y_L)_{\tilde{\Gamma}} \cong \mathcal{I}/(\mathcal{V} + \mathcal{I}^2),$$

where $\mathcal{V}$ is the $\mathbf{Z}_p$-module generated by $U_p - 1$. By definition, $\mathcal{P}_{E/F}^{(1)}$ contains the image of $(\mathcal{P}_{L/K})_{\tilde{\Gamma}}$ in $(I_Q/I_Q^2 \otimes A_F/p)^{(1)}$. In the proof of Theorem 5.2 it was shown that $\mathcal{U}_K^{(1)} = \mathcal{U}_{L/K}^{(1)}$, and hence the element p in F is a universal norm from L. Therefore, $\mathcal{P}_{E/F}^{(1)}$ is in fact equal to the image of $(\mathcal{P}_{L/K})_{\tilde{\Gamma}}$. It follows that we have an isomorphism in the quotient

$$(I_Q/I_Q^2 \otimes A_F/p)^{(1)}/\mathcal{P}_{E/F}^{(1)} \cong \mathcal{I}/(\mathcal{V} + \mathfrak{m}\mathcal{I}). \qquad (5.4)$$

Now, $\langle p, \eta \rangle_{1,F}$ is nontrivial if and only if $\mathcal{P}_{E/F}^{(1)}$ is nontrivial as well. By (5.2) and (5.4), the latter nontriviality occurs if and only if the image of $U_p - 1$ in $\mathcal{I}/\mathfrak{m}\mathcal{I}$ is nonzero. $\qquad \square$

We now focus on the case that $N = 1$, so $F = \mathbf{Q}(\mu_p)$, $K = \mathbf{Q}(\mu_{p^\infty})$, and $\theta = \omega^k$ for some positive even integer $k < p$ with $p \mid B_k$ and $p \nmid B_{p+1-k}$. In general, we call (p, k) an irregular pair if k is a positive even integer such that $k < p$ and $p \mid B_k$.

The following was proven using a program we wrote for the computational algebra package Magma. In particular, we used its built-in routines for modular symbols (programmed by W. Stein).

Theorem 5.7. *For all irregular pairs (p, k) with $p < 1000$, the element $U_p - 1$ generates $\mathcal{I}_2$.*

Sketch of Proof. It suffices to show that $U_p - 1$ generates the abelian group $\mathcal{I}_2/\mathcal{I}_2^2$. We did this by computation as follows. Let $\mathcal{H}_2$ denote the full cuspidal Hecke algebra over $\mathbf{Z}$ of weight 2, level p, and character ω^{k-2}. We computed the Hecke operators U_p and T_i in $\mathcal{H}_2$ for all i with $1 \leq i \leq (p+1)/6$ as elements of $M_d(\mathbf{Z}[\xi])$ for an appropriate d and a fixed primitive $(p-1)$st root of unity ξ, which act on the space of cuspidal modular symbols with respect to a fixed choice of basis. These T_i generate $\mathcal{H}_2$ as a $\mathbf{Z}$-submodule (for the case of $\Gamma_0(p)$, see [AS], where the result is derived from a result of J. Sturm; the case of $\Gamma_1(p)$ and arbitrary character follows similarly from [St, Corollary 9.20]). We then considered the images $\bar{T}_i$ and $\bar{U}_p$ of these matrices in $M_d(\mathbf{Z}/p^2\mathbf{Z})$, via the appropriate choice of embedding of $\mathbf{Z}[\xi]$ in $\mathbf{Z}_p$ followed by reduction.

Let M denote the subgroup of $M_d(\mathbf{Z}/p^2\mathbf{Z})$ spanned by the $\bar{T}_i$. We checked that pM has p-rank d, the rank of $\mathcal{H}_2$ as a $\mathbf{Z}$-module. We then computed the minimal integer N such that the $\bar{T}_i$ with $1 \leq i \leq N$ generate M as a $\mathbf{Z}/p^2\mathbf{Z}$-module. Then p and the $T_i - \sum_{0 < e | i} \omega(e)^{k-2} e$ with $1 \leq i \leq N$ generate $\mathcal{I}_2$ over $\mathbf{Z}_p$, provided that

$$(\mathcal{H}_2 \otimes \mathbf{Z}_p)/\mathcal{I}_2 \cong \mathfrak{h}_2/\mathcal{I}_2 \cong \mathbf{Z}_p/B_{2,\omega^{k-2}}$$

is of order p, which we checked. We computed the images I and J of $\mathcal{I}_2$ and $\mathcal{I}_2^2$, respectively, in M using these elements and their products. We verified that I is generated by J and $\bar{U}_p - 1$. Since $\mathcal{I}_2^2$ contains $p^2 \mathfrak{h}_2$, this suffices to prove that $\mathcal{I}_2$ is generated by $\mathcal{I}_2^2$ and $U_p - 1$. □

We remark that for all irregular pairs (p, k) with $p < 1000$, we have $p \nmid B_{p+1-k}$. The following are then direct consequences of Theorems 5.6 and 5.7.

Corollary 5.8. *For all irregular primes p with $p < 1000$, we have the equality*

$$\langle p, \mathcal{U}_K \rangle_K = X_K(1).$$

Corollary 5.9. *For all $p < 1000$, the pairing $\langle \cdot, \cdot \rangle_{1,F}$ is surjective, and $\langle p, 1 - \zeta_p \rangle_{1,F}$ generates its image as a $\mathbf{Z}_p[\Delta]$-module.*

Thus, we have proven Theorem 1.2 of the introduction. Furthermore, $\langle \cdot, \cdot \rangle_{1,F}$ is given in each eigenspace, up to a scalar which is nonzero modulo p, by the formula that can now be found at `www.math.mcmaster.ca/~sharifi` (see [McS, Theorem 5.1]). This computation has numerous interesting corollaries, in particular to the structure of unramified pro-p Galois groups and to products in the K-theory of cyclotomic integer rings (including $\mathbf{Z}$), which we intend to discuss in later work. For now, we refer the reader to [McS] for the details of some of these applications and elaborate on just one.

The surjectivity of the cup product for $p = 691$, which divides B_{12}, implies a conjecture of Ihara's on the existence of a particular relation in the 12th graded piece of Ihara's $\mathbf{Z}_{691}$-Lie algebra arising from the outer Galois action on the pro-691 fundamental group $\pi^{[691]}$ of $\mathbf{P}^1 - \{0, 1, \infty\}$. We describe this conjecture briefly and refer the reader to [I] for more details. Specifically, if $\pi^{[p]}(i)$ denotes the ith term in the lower central series of $\pi^{[p]}$, one may consider the filtration of $G_{\mathbf{Q}}$ by the kernels $F^i G_{\mathbf{Q}}$ of the homomorphisms $G_{\mathbf{Q}} \to \mathrm{Out}(\pi^{[p]}/\pi^{[p]}(i+1))$ induced by the outer action of $G_{\mathbf{Q}}$ on $\pi^{[p]}$. The direct sum

$$\mathfrak{g}_p = \bigoplus_{i=1}^{\infty} F^i G_{\mathbf{Q}}/F^{i+1} G_{\mathbf{Q}}$$

forms a graded $\mathbf{Z}_p$-Lie algebra with commutator induced from that of $G_{\mathbf{Q}}$. Ihara showed that the Lie algebra $\mathfrak{g}_p$ contains nontrivial elements $\sigma_i \in \mathrm{gr}^i \mathfrak{g}_p$ for odd $i \geq 3$. P. Deligne conjectured that the graded $\mathbf{Q}_p$-Lie algebra $\mathfrak{g}_p \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is freely generated by the σ_i. On the other hand, Ihara conjectured a specific relation in $\mathfrak{g}_{691}$:

$$2[\sigma_3, \sigma_9] - 27[\sigma_5, \sigma_7] \in 691 \mathrm{gr}^{12} \mathfrak{g}_{691}.$$

This is an immediate corollary of Corollary 5.9 for $p = 691$ and [McS, Theorem 9.11]. The point is that this relation is the image of a relation in the Galois group of the maximal pro-691 unramified outside 691 extension of $\mathbf{Q}(\mu_{691})$ in which the powers of the commutators appearing in the relation are determined by values of the cup product on cyclotomic 691-units.

6 Selmer groups

In this section, we give one special example of the use of Theorem 3.5 in the computation of Selmer groups. That is, we consider Greenberg's Selmer group attached to Ohta's Galois module $\mathfrak{X}$ introduced in Section 2. We consider only the lattice $\mathfrak{X}$ in the tensor product of $\mathfrak{X}$ with the quotient field of the cuspidal Hecke algebra $\mathfrak{h}$, and we do not introduce any cyclotomic twist. (Another particularly nice choice is the lattice $\mathfrak{X}^+ \oplus \mathcal{I}\mathfrak{X}^-$ twisted by the inverse of $\det \rho$.)

Set $W = \mathfrak{X} \otimes_{\mathfrak{h}} \mathrm{Hom}_{\mathbf{Z}_p}(\mathfrak{h}/\mathcal{I}, \mathbf{Q}_p/\mathbf{Z}_p)$. This is isomorphic as a Hecke module to the Pontryagin dual of $\mathfrak{X}/\mathcal{I}\mathfrak{X}$, while its Galois action is induced by that on $\mathfrak{X}$. Recall our fixed decomposition and inertia groups D_p and I_p, respectively, at p in $G_{\mathbf{Q}}$. We will compute the Selmer group

$$\mathrm{Sel}(\mathbf{Q}, W) = \ker(H^1(G_{\mathbf{Q},S}, W) \to H^1(I_p, W))$$

and the strict Selmer group

$$\mathrm{Sel}_S(\mathbf{Q}, W) = \ker(H^1(G_{\mathbf{Q},S}, W) \to H^1(D_p, W)).$$

Such Selmer groups are defined and discussed in detail in [G]. (In general, it is necessary to take a quotient of W by a D_p-submodule in the local Selmer groups, but here that submodule is trivial as the Hodge-Tate weights specialize to 0 and $1 - k$ in weight k.)

We continue with the notation and hypotheses of the previous sections. The decomposition of $\mathfrak{X}$ as $\mathfrak{X}^+ \oplus \mathfrak{X}^-$ yields a decomposition of W into plus and minus parts which are $(\mathfrak{h}/\mathcal{I})$-modules. Note that $G_{\mathbf{Q}}$ acts by ϕ_C of (3.1) on W. In other words, W fits in an exact sequence

$$0 \to W^+ \to W \to W/W^+ \to 0$$

of Galois modules, where W^+ is fixed by $G_{\mathbf{Q}}$ and $W/W^+ = (W/W^+)^{(\omega\theta^{-1})}$. We have Selmer groups for W^+ and W/W^+ defined analogously to those of W.

We prove our result through a series of lemmas.

Lemma 6.1. *The Selmer group* $\mathrm{Sel}(\mathbf{Q}, W^+)$ *is trivial.*

Proof. We have an exact sequence

$$0 \to H^1(\tilde{\Gamma}, W^+) \to H^1(G_{\mathbf{Q},S}, W^+) \to \mathrm{Hom}_{\tilde{\Lambda}}(Z_K, W^+)$$

(where Z_K is the Galois group of the maximal abelian pro-p unramified outside p extension of K), and the rightmost group is trivial since W^+ has a trivial $\tilde{\Gamma}$-action. On the other hand, the facts that $p \nmid [F : \mathbf{Q}]$ and K/F is totally ramified at primes above p imply that $H^1(\tilde{\Gamma}, W^+)$ has trivial intersection with $\mathrm{Sel}(\mathbf{Q}, W^+)$, which must then be 0. $\qquad\square$

Lemma 6.2. *The natural map*

$$\mathrm{Sel}(\mathbf{Q}, W) \to \mathrm{Sel}(\mathbf{Q}, W/W^+)$$

is an isomorphism.

Proof. We remark that $H^2(G_{\mathbf{Q},S}, W^+)$ is trivial since W^+ is fixed by $\tilde{\Gamma}$. As $H^2(G_{\mathbf{Q},S}, W^+)$ is p-power torsion and $G_{\mathbf{Q},S}$ has p-cohomological dimension 2, this follows from the triviality of

$$H^2(G_{\mathbf{Q},S}, \mathbf{Z}/p\mathbf{Z}) \cong (A_{\mathbf{Q}(\mu_p)}/p)^{(\omega)} = 0.$$

Noting also that $(W/W^+)^{(1)} = 0$, we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(G_{\mathbf{Q},S}, W^+) & \longrightarrow & H^1(G_{\mathbf{Q},S}, W) & \longrightarrow & H^1(G_{\mathbf{Q},S}, W/W^+) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\phi} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(I_p, W^+) & \longrightarrow & H^1(I_p, W) & \longrightarrow & H^1(I_p, W/W^+). & &
\end{array}
$$

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In fact, ϕ factors through $H^1(D_p, W^+)$, and hence its image is contained in the D_p/I_p-invariant part of $H^1(I_p, W^+)$. Let ϕ' denote the map to this group induced by ϕ. By the Snake Lemma and Lemma 6.1, we obtain an exact sequence

$$0 \to \mathrm{Sel}(\mathbf{Q}, W) \to \mathrm{Sel}(\mathbf{Q}, W/W^+) \to \mathrm{coker}\,\phi'.$$

We remark that

$$H^1(I_p, W^+)^{D_p/I_p} \cong H^1(\tilde{\Gamma}, W^+),$$

which shows that ϕ' is surjective. $\qquad\square$

Lemma 6.3. *There is a canonical isomorphism of $\mathbf{Z}_p$-modules*

$$\mathrm{Sel}(\mathbf{Q}, W/W^+) \xrightarrow{\sim} \mathrm{Hom}_\mathfrak{h}(\mathcal{I}/\mathcal{I}^2, W^+),$$

which is given by lifting to $H^1(G_{\mathbf{Q},S}, W)$ followed by restriction to $G_{HL,S}$. The resulting homomorphism has image in W^+ and factors through $(I_G X_L / I_G^2 X_L)_{\tilde{\Gamma}} \cong \mathcal{I}/\mathcal{I}^2$, the latter isomorphism being as in Theorem 3.5.

Proof. Since $(W/W^+)^{(1)} = 0$, we have an isomorphism

$$H^1(G_{\mathbf{Q},S}, W/W^+) \cong H^1(G_{K,S}, W/W^+)^{\tilde{\Gamma}}.$$

Hence, we have

$$\mathrm{Sel}(\mathbf{Q}, W/W^+) \cong \mathrm{Hom}_{\tilde{\Lambda}}(X_K, W/W^+).$$

Eigenspace considerations force an isomorphism

$$\mathrm{Hom}_{\tilde{\Lambda}}(X_K, W/W^+) \cong \mathrm{Hom}_\Lambda(X_K^{(\omega\theta^{-1})}, W/W^+).$$

The latter group is computable by Theorem 2.1:

$$\mathrm{Hom}_\Lambda(X_K^{(\omega\theta^{-1})}, W/W^+) \cong \mathrm{Hom}_\mathfrak{h}(B/\mathcal{I}B, W^-).$$

Since $C/\mathcal{I}C$ is free of rank 1 over $\mathfrak{h}/\mathcal{I}$, we have that

$$\mathrm{Hom}_\mathfrak{h}(B/\mathcal{I}B, W^-) \cong \mathrm{Hom}_\mathfrak{h}(B/\mathcal{I}B \otimes_\mathfrak{h} C/\mathcal{I}C, W^- \otimes_\mathfrak{h} C/\mathcal{I}C). \tag{6.1}$$

Since $C = \mathrm{End}_\mathfrak{h}(\mathfrak{X}^-, \mathfrak{X}^+) \cong \mathfrak{h}$, we have a canonical isomorphism

$$W^- \otimes_\mathfrak{h} C/\mathcal{I}C \cong W^+.$$

This and the fact that $B/\mathcal{I}B \otimes_\mathfrak{h} C/\mathcal{I}C \cong \mathcal{I}/\mathcal{I}^2$ now imply the existence of the desired isomorphism.

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Let us show that this isomorphism agrees with its description in the statement of this theorem. That is, take a cocycle f with class in $\mathrm{Sel}(\mathbf{Q}, W/W^+)$, and let $\tilde{f}$ be a cocycle with class in $H^1(G_{\mathbf{Q},S}, W)$ lifting that of f. Then the restriction $\tilde{f}|_{G_{K,S}}$ lifts the homomorphism $f|_{G_{K,S}}$. Furthermore, $\tilde{f}|_{G_{HL,S}}$ is a homomorphism with image in W^+. Let E denote the fixed field of the kernel of $\tilde{f}|_{G_{HL,S}}$, so that $\tilde{f}|_{G_{HL,S}}$ factors through a map g on $\mathrm{Gal}(E/HL)$. Since no $\tilde{\Delta}$-invariant homomorphisms $G_{K,S} \to \mathbf{Z}/p\mathbf{Z}$ exist, $\mathrm{Gal}(E/HL)$ is contained in the commutator subgroup of $\mathrm{Gal}(E/K)$. As $G_{L,S}$ acts trivially on W, we have

$$\tilde{f}([\sigma, \tau]) = (\sigma - 1)\tilde{f}(\tau) = \bar{c}(\sigma)f(\tau) \in W^+.$$

for $\sigma \in G_{K,S}$ and $\tau \in G_{L,S}$. Therefore, the map which takes f to g is, by the identifications in Theorem 3.5, both the isomorphism in (6.1) and given by the description in the statement of the lemma. $\qquad \square$

Lemmas 6.2 and 6.3 yield the following as an immediate corollary.

Theorem 6.4. *There is a canonical isomorphism of $\mathbf{Z}_p$-modules*

$$\mathrm{Sel}(\mathbf{Q}, W) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{h}}(\mathcal{I}/\mathcal{I}^2, W^+).$$

As for the strict Selmer group, we obtain the following. Recall that $\mathcal{V}$ denotes the $\mathbf{Z}_p$-module generated by $U_p - 1$.

Theorem 6.5. *There is a canonical isomorphism of $\mathbf{Z}_p$-modules*

$$\mathrm{Sel}_S(\mathbf{Q}, W) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{h}}(\mathcal{I}/(\mathcal{V} + \mathcal{I}^2), W^+).$$

Proof. Let ϑ denote the homomorphism

$$\vartheta \colon H^1(G_{\mathbf{Q},S}, W^+) \to H^1(D_p, W^+).$$

Again, $\mathrm{Sel}_S(\mathbf{Q}, W^+) = 0$, and as in the proof of Theorem 6.4, we obtain an exact sequence

$$0 \to \mathrm{Sel}_S(\mathbf{Q}, W) \to \mathrm{Sel}_S(\mathbf{Q}, W/W^+) \xrightarrow{\delta} \mathrm{coker}\,\vartheta.$$

Note that $\mathrm{Sel}_S(\mathbf{Q}, W/W^+) \cong \mathrm{Sel}(\mathbf{Q}, W/W^+)$, as $Y_K^{(\omega\theta^{-1})} \cong X_K^{(\omega\theta^{-1})}$. Since D_p acts trivially on W^+, we have

$$H^1(D_p, W^+) \cong H^1(\tilde{\Gamma}, W^+) \oplus H^1(D_p/I_p, W^+),$$

and the cokernel of ϑ is canonically isomorphic to $H^1(D_p/I_p, W^+)$.

By Lemma 6.3 and the Snake Lemma, δ induces a map

$$\operatorname{Hom}_{\mathfrak{h}}(\mathcal{I}/\mathcal{I}^2, W^+) \to H^1(D_p/I_p, W^+),$$

which is given by identifying $\mathcal{I}/\mathcal{I}^2$ and $(I_G X_L/I_G^2 X_L)_{\tilde{\Gamma}}$ as in Theorem 3.5, restricting to the image of $D_p \cap G_{L,S}$ in this group, which is isomorphic to a quotient of D_p/I_p, and finally inflating. By Theorem 3.6, the kernel of this map is canonically isomorphic to $\operatorname{Hom}_{\mathfrak{h}}(\mathcal{I}/(\mathcal{V} + \mathcal{I}^2), W^+)$, as desired. $\qquad\square$

Let $\mathfrak{W} = \mathfrak{X} \otimes_{\mathfrak{h}} \operatorname{Hom}_{\mathbf{Z}_p}(\mathfrak{h}, \mathbf{Q}_p/\mathbf{Z}_p)$. We obtain as a corollary the following small piece of a "main conjecture" for an Eisenstein component of the Hida representation, which should be a direct analogue of [G, Conjecture 2.2]. Note that $U_p - 1$ occurs as a factor in the two-variable p-adic L-function of Mazur and K. Kitagawa at the trivial character (see [Ki, Theorem 4.8], for instance).

Proposition 6.6. *If $\mathcal{I}/\mathcal{I}^2$ is generated as a $\mathbf{Z}_p$-module by $U_p - 1$, then $\operatorname{Sel}_S(\mathbf{Q}, \mathfrak{W}) = 0$.*

Proof. Since $\mathcal{I}$ is a principal ideal of $\mathfrak{h}$ generated by $U_p - 1$, we have an exact sequence

$$0 \to W \to \mathfrak{W} \xrightarrow{\cdot(U_p-1)} \mathfrak{W} \to 0.$$

Since the image of $c|_{I_p}$ generates $\mathfrak{h}$ and $d|_{I_p}$ is trivial, we have $\mathfrak{W}^{G_{\mathbf{Q},S}} \subseteq \mathfrak{W}^+$ and $\mathfrak{W}^{D_p} \subseteq \mathfrak{W}^+$. Since d has image U_p on the inverse of a Frobenius at p in D_p and $U_p - 1$ generates $\mathcal{I}$, we have furthermore that $\mathfrak{W}^{G_{\mathbf{Q},S}}$ and $\mathfrak{W}^{D_p}$ are contained in W^+. Finally, since the images of b and $d - 1$ are contained in $\mathcal{I}$, we have equality, i.e.,

$$\mathfrak{W}^{G_{\mathbf{Q},S}} = \mathfrak{W}^{D_p} = W^+.$$

A diagram chase then provides an exact sequence of Selmer groups

$$\operatorname{Sel}_S(\mathbf{Q}, W) \to \operatorname{Sel}_S(\mathbf{Q}, \mathfrak{W}) \xrightarrow{\cdot(U_p-1)} \operatorname{Sel}_S(\mathbf{Q}, \mathfrak{W}).$$

By Theorem 6.5, $\operatorname{Sel}_S(\mathbf{Q}, W) = 0$, and the result follows from Nakayama's Lemma applied to the finitely generated $\mathfrak{h}$-module $\operatorname{Hom}_{\mathbf{Z}_p}(\operatorname{Sel}_S(\mathbf{Q}, \mathfrak{W}), \mathbf{Q}_p/\mathbf{Z}_p)$. $\qquad\square$

A   Eigenspaces

In this appendix, we prove a decomposition result (Proposition A.2) for the structure of eigenspaces of tensor products of $\mathbf{Z}_p[\tilde{\Delta}]$-modules. We use only its consequence (Lemma A.3) for trivial eigenspaces in this article, but the general result would be useful should one wish to consider more general eigenspaces.

Let $\tilde{\Delta}$ be a finite abelian group of order prime to p. Let $\tilde{\Delta}^*$ denote the group of $\overline{\mathbf{Q}_p}$-valued characters on $\tilde{\Delta}$. Given a character $\psi \in \tilde{\Delta}^*$, we let R_ψ denote the unramified extension of $\mathbf{Z}_p$ generated by the values of ψ. We denote the set of $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$-conjugacy classes of characters in $\tilde{\Delta}^*$ by Σ, and we denote the conjugacy class of ψ by $[\psi]$. We have a ring decomposition (see [MW, Section 1.3])

$$\mathbf{Z}_p[\tilde{\Delta}] \cong \prod_{[\psi] \in \Sigma} R_\psi.$$

In particular, the surjection $\mathbf{Z}_p[\tilde{\Delta}] \to R_\psi$ induced by ψ is split.

If A is $\mathbf{Z}_p[\tilde{\Delta}]$-module, then we let

$$A^{(\psi)} = A \otimes_{\mathbf{Z}_p[\tilde{\Delta}]} R_\psi \cong \{a \in A \otimes_{\mathbf{Z}_p} R_\psi \mid \delta a = \psi(\delta)a \text{ for all } \delta \in \tilde{\Delta}\},$$

where the map $\mathbf{Z}_p[\tilde{\Delta}] \to R_\psi$ in the tensor product is the surjection induced by ψ. We remark that $A^{(\psi)}$ inherits the structure of a $R_\psi[\tilde{\Delta}]$-module. Again, we have a decomposition

$$A \cong \bigoplus_{[\psi] \in \Sigma} A^{(\psi)}. \tag{A.1}$$

In particular $A^{(\psi)}$ and $A^{(\chi)}$ are equal as subgroups and isomorphic as $\mathbf{Z}_p[\tilde{\Delta}]$-submodules of A if $[\psi] = [\chi]$.

Let $\mathcal{R}$ denote the subring of $\overline{\mathbf{Q}_p}$ generated by all character values of $\tilde{\Delta}$ over $\mathbf{Z}_p$. Let $A^{\mathcal{R}} = A \otimes_{\mathbf{Z}_p} \mathcal{R}$, an $\mathcal{R}[\tilde{\Delta}]$-module. We set

$$A^\psi = A^{\mathcal{R}} \otimes_{\mathcal{R}[\tilde{\Delta}]} \mathcal{R} \cong A \otimes_{\mathbf{Z}_p[\tilde{\Delta}]} \mathcal{R} \cong A^{(\psi)} \otimes_{R_\psi} \mathcal{R},$$

where $\mathbf{Z}_p[\tilde{\Delta}] \to \mathcal{R}$ and $\mathcal{R}[\tilde{\Delta}] \to \mathcal{R}$ are the maps induced by ψ and $R_\psi \to \mathcal{R}$ is the natural inclusion. After extending scalars to $\mathcal{R}$, our decomposition takes the simpler form

$$A^{\mathcal{R}} \cong \bigoplus_{\chi \in \tilde{\Delta}^*} A^\chi.$$

We wish to work with eigenspaces of tensor products. Let us begin by proving such a result after extension of scalars.

Lemma A.1. *Let $\psi \in \tilde{\Delta}^*$, and let A and B be $\mathbf{Z}_p[\tilde{\Delta}]$-modules. Then*

$$(A \otimes_{\mathbf{Z}_p} B)^\psi \cong \bigoplus_{\chi \in \tilde{\Delta}^*} (A^\chi \otimes_{\mathcal{R}} B^{\psi\chi^{-1}}).$$

Proof. Note that

$$A \otimes_{\mathbf{Z}_p} B \otimes_{\mathbf{Z}_p} \mathcal{R} \cong A^{\mathcal{R}} \otimes_{\mathcal{R}} B^{\mathcal{R}}.$$

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Decomposing $A^{\mathcal{R}}$ and $B^{\mathcal{R}}$, we have that

$$A^{\mathcal{R}} \otimes_{\mathcal{R}} B^{\mathcal{R}} \cong (\bigoplus_{\chi \in \tilde{\Delta}^*} A^{\chi}) \otimes_{\mathcal{R}} (\bigoplus_{\varphi \in \tilde{\Delta}^*} B^{\varphi}) \cong \bigoplus_{\chi, \varphi \in \tilde{\Delta}^*} (A^{\chi} \otimes_{\mathcal{R}} B^{\varphi \chi^{-1}}).$$

$\square$

We now prove our decomposition result for eigenspaces of tensor products.

Proposition A.2. *Let A and B be $\mathbf{Z}_p[\tilde{\Delta}]$-modules, and let $\varphi \in \tilde{\Delta}^*$. Let*

$$\Sigma_{\varphi}^2 = \{([\chi], [\psi]) \in \Sigma \times \Sigma \mid \exists \theta \in [\chi] \text{ with } \varphi\theta^{-1} \in [\psi]\}.$$

Then we have a decomposition of $\mathbf{Z}_p[\tilde{\Delta}]$-modules

$$(A \otimes_{\mathbf{Z}_p} B)^{(\varphi)} \cong \bigoplus_{([\chi],[\psi]) \in \Sigma_{\varphi}^2} (A^{(\chi)} \otimes_{\mathbf{Z}_p} B^{(\psi)})^{(\varphi)}.$$

Proof. Since A and B both decompose into eigenspaces as in (A.1), it suffices to show that

$$(A^{(\chi)} \otimes_{\mathbf{Z}_p} B^{(\psi)})^{(\varphi)} = 0$$

for $\chi, \psi \in \tilde{\Delta}^*$ with $([\chi], [\psi]) \notin \Sigma_{\varphi}^2$. Note that $(A^{(\chi)})^{\theta} = 0$ for $\theta \in \tilde{\Delta}^*$ unless $\theta \in [\chi]$, in which case $(A^{(\chi)})^{\theta} \cong A^{\theta}$. Therefore, Lemma A.1 implies that

$$(A^{(\chi)} \otimes_{\mathbf{Z}_p} B^{(\psi)})^{\varphi} \cong \bigoplus_{\theta \in [\chi]} A^{\theta} \otimes_{\mathcal{R}} (B^{(\psi)})^{\varphi\theta^{-1}},$$

which is trivial since $\varphi\theta^{-1} \notin [\psi]$ for all $\theta \in [\chi]$. $\square$

We have the following result for the special case of the trivial eigenspace of the tensor product.

Lemma A.3. *Let A and B be $\mathbf{Z}_p[\tilde{\Delta}]$-modules. Then*

$$(A^{(\chi)} \otimes_{\mathbf{Z}_p} B)^{(1)} \cong (A^{(\chi)} \otimes_{\mathbf{Z}_p} B^{(\chi^{-1})})^{(1)} \cong A^{(\chi)} \otimes_{R_{\chi}} B^{(\chi^{-1})}.$$

Proof. Note that $\Sigma_1^2 = \{([\chi], [\chi^{-1}]) \mid [\chi] \in \Sigma\}$. The result then follows from Proposition A.2 and the isomorphisms

$$(A^{(\chi)} \otimes_{\mathbf{Z}_p} B^{(\chi^{-1})})^{(1)} \cong \frac{A^{(\chi)} \otimes_{\mathbf{Z}_p} B^{(\chi^{-1})}}{\langle \chi(\delta)a \otimes b - a \otimes \chi(\delta)b \mid a, b, \delta \rangle} \cong A^{(\chi)} \otimes_{R_{\chi}} B^{(\chi^{-1})},$$

(where a, b, and δ in the quotient run over elements of $A^{(\chi)}$, $B^{(\chi^{-1})}$ and $\tilde{\Delta}$, respectively). $\square$

References

[AS]   A. Agashe and W. Stein, Appendix to J.-C. Lario and R. Schoof, *Some computations with Hecke rings and deformation rings*, Experiment. Math. **11** (2002), 303–311.

[FW]   B. Ferrero and L. Washington, *The Iwasawa invariant μ_p vanishes for abelian number fields*, Ann. of Math. (2) **109** (1979), 377–395.

[G]   R. Greenberg, "Iwasawa theory and p-adic deformations of motives" in *Motives (Seattle, WA, 1991)*, Proc. Sympos. Pure Math. **55**, Part 2, Amer. Math. Soc., Providence, 1994, 193–223.

[HP]   G. Harder and R. Pink, *Modular konstruierte unverzweigte abelsche p-Erweiterungen von $\mathbf{Q}(\mu_p)$ und die Struktur ihrer Galoisgruppen*, Math. Nachr. **159** (1992), 83–99.

[H1]   H. Hida, *Galois representations into $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms*, Invent. Math. **85** (1986), 545–613.

[H2]   H. Hida, *Iwasawa modules attached to congruences of cusp forms*, Ann. Sci. École Norm. Sup. (4) **19** (1986), 231–273.

[I]   Y. Ihara, "Some arithmetic aspects of Galois actions on the pro-p fundamental group of $\mathbb{P}^1-\{0,1,\infty\}$" in *Arithmetic Fundamental Groups and Noncommutative Algebra (Berkeley, CA, 1999)*, ed. M. D. Fried and Y. Ihara, Proc. Sympos. Pure Math. **70**, Amer. Math. Soc., Providence, 2002, 247–273.

[Ki]   K. Kitagawa, "On standard p-adic L-functions of families of elliptic cusp forms" in *p-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991)*, Contemp. Math. **165**, Amer. Math. Soc., Providence, 1994, 81–110.

[Ku]   M. Kurihara, *Ideal class groups of cyclotomic fields and modular forms of level 1*, J. Number Theory **45** (1993), 281–294.

[L]   S. Lang, *Cyclotomic fields I and II*, combined 2nd ed., Grad. Texts in Math. **121**, Springer, New York, 1990.

[MW]   B. Mazur and A. Wiles, *Class fields of abelian extensions of $\mathbf{Q}$*, Invent. Math. **76** (1984), 179–330.

[McS]   W. McCallum and R. Sharifi, *A cup product in the Galois cohomology of number fields*, Duke Math. J. **120** (2003), 269–310.

[O1]   M. Ohta, *On the p-adic Eichler-Shimura isomorphism for Λ-adic cusp forms*, J. reine angew. Math. **463** (1995), 49–98.

[O2]   M. Ohta, *Ordinary p-adic étale cohomology groups attached to towers of elliptic modular curves*, Compositio Math. **115** (1999), 241–301.

[O3]   M. Ohta, *Ordinary p-adic étale cohomology groups attached to towers of elliptic modular curves. II*, Math. Ann. **318** (2000), 557–583.

[O4]   M. Ohta, *Congruence modules attached to Eisenstein series*, Ann. Sci. École Norm. Sup. (4) **36** (2003), 225–269.

[O5]   M. Ohta, *Companion forms and the structure of p-adic Hecke algebras*, J. reine angew. Math. **585** (2005), 141–172.

[Ri]   K. Ribet, *A modular construction of unramified p-extensions of* $\mathbf{Q}(\mu_p)$, Invent. Math. **34** (1976), 151–162.

[Sh1]   R. Sharifi, *Determination of conductors from Galois module structure*, Math. Z. **241** (2002), 227–245.

[Sh2]   R. Sharifi, *Massey products and ideal class groups*, J. reine angew. Math. **603** (2007), 1–33.

[Sh3]   R. Sharifi, Appendix to Y. Hachimori and R. Sharifi, *On the failure of pseudo-nullity of Iwasawa modules*, J. Alg. Geom. **14** (2005), 567–591.

[Si]   W. Sinnott, Appendix to L. Federer and B. Gross, *Regulators and Iwasawa modules*, Invent. Math. **62** (1981), 443–458.

[St]   W. Stein, *Modular forms: a computational approach*, Grad. Stud. in Math. **79**, Amer. Math. Soc., Providence, 2007.

[SW]   C. Skinner and A. Wiles, *Ordinary representations and modular forms*, Proc. Nat. Acad. Sci. USA **94** (1997), 10520–10527.

[So]   C. Soulé, "On higher *p*-adic regulators" in *Algebraic K-theory (Evanston, Ill., 1980)*, Lecture Notes in Math. **854**, Springer, Berlin, 1981, 372–401.

[Wa]   L. Washington, *Introduction to cyclotomic fields*, 2nd ed., Springer, New York, 1997.

[Wi]   A. Wiles, *The Iwasawa conjecture for totally real fields*, Ann. of Math. (2) **131** (1990), 493–540.

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