

# An extension of the Fukaya-Kato method

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## Abstract

In the groundbreaking paper [FK], T. Fukaya and K. Kato proved a slight weakening of a conjecture of the author's [Sh1] under an assumption that a Kubota-Leopoldt  $p$ -adic  $L$ -function has no multiple zeros. This article describes a refinement of their method that sheds light on the role of the  $p$ -adic  $L$ -function.

## 1 Introduction

In this paper, we explore the conjectural relationship between

- modular symbols in the quotient  $P$  of the real part of the first homology group of a modular curve of level  $N$  by the action of an Eisenstein ideal, and
- cup products of cyclotomic units in a second Galois cohomology group  $Y$  of the cyclotomic field  $\mathbb{Q}(\mu_N)$  with restricted ramification,

More precisely, we consider maximal quotients of  $p$ -parts of the latter groups for an odd prime  $p$  dividing  $N$  on which  $(\mathbb{Z}/N\mathbb{Z})^\times$  acts through a given even character  $\theta$  via diamond operators and Galois elements, respectively. In [Sh1], we constructed two maps  $\varpi: P \rightarrow Y$  and  $\Upsilon: Y \rightarrow P$  and conjectured them to be inverse to each other, up to a canonical unit suspected to be 1 (see Conjecture 3.1.9). The map  $\varpi$  was defined explicitly to take a modular symbol to a cyclotomic unit, while  $\Upsilon$  was defined through the Galois action on the homology of a modular curve, or a tower thereof, in the spirit of the Mazur-Wiles method of proof of the main conjecture. By the main conjecture, both the homology group and the Galois cohomology group in question are annihilated by a power series  $\xi$  corresponding to a  $p$ -adic  $L$ -function. This power series  $\xi$  is (roughly) both a generator of the characteristic ideal of inverse limit of

Galois cohomology groups up the  $p$ -cyclotomic tower and the constant term from an ordinary family of Eisenstein series for  $\theta$ .

Fukaya and Kato showed in [FK] that  $\xi'Y \circ \varpi = \xi'$  modulo torsion in  $P$ , where  $\xi'$  is essentially the derivative of  $\xi$ . In Theorem 5.3.8, we show that this identity holds in  $P$  itself, employing joint work from [FKS2]. At least up to finite torsion in  $P$ , the conjecture then follows if  $\xi'$  happens to be relatively prime to  $\xi$  in the relevant Iwasawa algebra.

Considerable progress has been made in the study of  $Y$  by Wake and Wang-Erickson [WWE] and Ohta [Oh3], by different methods. In cases that  $Y$  is known to be an isomorphism and  $Y$  is pseudo-cyclic, the identity of Fukaya and Kato implies the original conjecture, i.e., up to unit. This pseudo-cyclicity was related to the question of localizations of Hecke algebras being Gorenstein in the work of Wake and Wang-Erickson, as well as to the question of  $Y$  being a pseudo-isomorphism. Ohta shows that  $Y$  is in fact an isomorphism under an assumption on the relevant Dirichlet character that holds in the case of trivial tame level. We note that this implies in particular that  $P$  has no torsion in such eigenspaces, as  $Y$  does not.

The pseudo-cyclicity of  $Y$  is expected to hold as a consequence of a well-known and widely believed conjecture of Greenberg's on the finiteness of the plus part of the unramified Iwasawa module. Moreover, since the  $p$ -adic  $L$ -functions in question are unlikely to ever have multiple zeros, one would expect the unit in our conjecture to always be 1, as in its stronger form. Nevertheless, this might appear to reduce the conjecture to chance, which is less than desirable. This motivates us to attempt a finer study.

Our aim in this paper is to study the role of  $\xi'$  in the work of Fukaya-Kato and ask whether it is possible to remove it in the method. As we shall see, this would be possible but for a global obstruction that stands in the way. We make this explicit by deducing an equivalent form of our conjecture in Theorem 5.5.1. The key idea is to consider cohomology groups which are intermediate between the restricted Galois cohomology of  $\mathbb{Q}$  and Iwasawa cohomology over the cyclotomic  $\mathbb{Z}_p$ -extension of  $\mathbb{Q}$  in an atypical sense. That is, via Shapiro's lemma, we may view Iwasawa cohomology as a cohomology group over  $\mathbb{Q}$  with coefficients in an induced module. We consider the cohomology of a quotient of these induced coefficients by an arithmetically relevant two-variable power series. In particular, the cohomology of this *intermediate quotient* (see Definition 4.2.1) is not the cohomology of any intermediate extension. Crucial to this work is a rather peculiar, but surprisingly clean and quite general, construction of intermediate Coleman maps in Section 4.2.

We also show that the global obstruction would vanish under a divisibility of Beilinson-Kato elements by one minus the  $p$ th Hecke operator at an intermediate stage between Iwasawa cohomology and cohomology at the ground level: see Question 5.5.2. This "intermediate

global divisibility” can be rephrased as the existence of a certain intermediate zeta map. The global obstruction to our conjecture corresponds to a weaker statement of existence of what would be a reduction of this map modulo the Eisenstein ideal. This reduced map can be characterized by properties of compatibility with a zeta map at the ground level of  $\mathbb{Q}$  and with a  $p$ -adically local version of the intermediate zeta map which we show does indeed exist.

Of course, this leaves us with the question of whether these intermediate zeta maps are likely to exist. As such, we perform a feasibility check for an analogue of the conditions of Theorem 5.5.1 in a simpler setting, with cyclotomic units in place of Beilinson-Kato elements. That is, in Section 6, we explore the analogues of global obstruction and divisibility for cohomology with coefficients in a Tate module, rather than the étale homology of a tower of modular curves. We show that the global obstruction in the cohomology of the intermediate quotient does in fact vanish in this setting, while verifying intermediate global divisibility only under an assumption of vanishing of a  $p$ -part of a class group of a totally real abelian field. This is in line with our suspicions that intermediate global divisibility may be too much to hope for in general, while still lending some credence to the conjecture that  $\Upsilon$  and  $\mathfrak{O}$  are indeed inverse maps, and not just by chance.

## 2 Background

In this section, we introduce many of our objects of study and known results on them.

### 2.1 Ordinary Hecke modules

**Definition 2.1.1.** Let  $p \geq 5$  be a prime and  $N \geq 4$  a positive integer with  $p \nmid N\varphi(N)$ , for  $\varphi$  the Euler-phi function.

Let  $\mathfrak{h}^{\text{ord}}$  denote Hida’s  $\mathbb{Z}_p$ -Hecke algebra acting on the space  $\mathfrak{S}^{\text{ord}}$  of ordinary “ $\Lambda$ -adic” cusp forms of level  $Np^\infty$ . Similarly, let  $\mathfrak{H}^{\text{ord}}$  denote Hida’s Hecke algebra acting on the space  $\mathfrak{M}^{\text{ord}}$  of ordinary  $\Lambda$ -adic forms of level  $Np^\infty$ . Note that  $\mathfrak{h}^{\text{ord}}$  is a quotient of  $\mathfrak{H}^{\text{ord}}$ .

In addition to  $\mathfrak{S}^{\text{ord}}$ , we have two related  $\mathfrak{h}^{\text{ord}}$ -modules.

**Definition 2.1.2.** The following inverse limits are taken with respect to trace maps.

- a. We let  $\mathcal{S}^{\text{ord}}$  denote the fixed part under complex conjugation (or “plus part”, denoted by the superscript “+”)

$$\mathcal{S}^{\text{ord}} = \varprojlim_r H_1(X_1(Np^r)(\mathbb{C}), \mathbb{Z}_p)^{\text{ord},+}$$

of the space of ordinary  $\Lambda$ -adic cuspidal modular symbols.

b. We let  $\mathcal{T}^{\text{ord}}$  denote the inverse limit

$$\mathcal{T}^{\text{ord}} = \varprojlim_r H_{\text{ét}}^1(X_1(Np^r)/\overline{\mathbb{Q}}, \mathbb{Z}_p(1))^{\text{ord}}$$

of ordinary parts of first étale cohomology groups of the closed modular curves  $X_1(Np^r)$ .

*Remark 2.1.3.* By viewing  $\overline{\mathbb{Q}}$  as the algebraic numbers in  $\mathbb{C}$ , we have an identification  $\mathcal{T}^{\text{ord},+} \cong \mathcal{S}^{\text{ord}}$  of  $\mathfrak{h}^{\text{ord}}$ -modules induced by the usual (i.e., complex) Eichler-Shimura isomorphisms at each stage of the modular tower. We note that Hecke actions on inverse limits of cohomology (as opposed to homology) groups are via the dual, or adjoint, operators.

Similarly but less crucially for our purposes, we have the following  $\mathfrak{H}^{\text{ord}}$ -modules.

**Definition 2.1.4.**

a. We let  $\mathcal{M}^{\text{ord}}$  denote the plus part

$$\mathcal{M}^{\text{ord}} = \varprojlim_r H_1(X_1(Np^r)(\mathbb{C}), \{\text{cusps}\}, \mathbb{Z}_p)^{\text{ord},+}$$

of the space of ordinary  $\Lambda$ -adic modular symbols.

b. We let  $\tilde{\mathcal{T}}^{\text{ord}}$  denote the inverse limit

$$\tilde{\mathcal{T}}^{\text{ord}} = \varprojlim_r H_{\text{ét}}^1(Y_1(Np^r)/\overline{\mathbb{Q}}, \mathbb{Z}_p(1))^{\text{ord}}$$

of ordinary parts of étale cohomology groups of open modular curves. Similarly, we let  $\tilde{\mathcal{T}}_c^{\text{ord}}$  denote the inverse limit of the ordinary parts of the compactly supported étale cohomology groups  $H_{c,\text{ét}}^1(Y_1(Np^r)/\overline{\mathbb{Q}}, \mathbb{Z}_p(1))$ .

As in the cuspidal case, the  $\mathfrak{H}^{\text{ord}}$ -modules  $\mathcal{M}^{\text{ord}}$  and  $\tilde{\mathcal{T}}^{\text{ord},+}$  are isomorphic.

*Remark 2.1.5.* Since signs are quite subtle in this work, we mention some conventions of algebraic topology used here and in [FK] (cf. [Ka, 2.7]), as well as some calculations which follow from them. Consider the compatible  $G_{\mathbb{Q}}$ -equivariant Poincaré duality pairings on étale cohomology:

$$\begin{aligned} H_{\text{ét}}^1(X_1(Np^r)/\overline{\mathbb{Q}}, \mathbb{Z}_p(1)) \times H_{\text{ét}}^1(X_1(Np^r)/\overline{\mathbb{Q}}, \mathbb{Z}_p(1)) &\xrightarrow{\cup} \mathbb{Z}_p(1), \\ H_{\text{ét}}^1(Y_1(Np^r)/\overline{\mathbb{Q}}, \mathbb{Z}_p(1)) \times H_{c,\text{ét}}^1(Y_1(Np^r)/\overline{\mathbb{Q}}, \mathbb{Z}_p(1)) &\xrightarrow{\cup} \mathbb{Z}_p(1). \end{aligned}$$

Viewing  $\overline{\mathbb{Q}}$  as the algebraic numbers in  $\mathbb{C}$ , these are compatible with the usual pairings of Poincaré duality for the isomorphic Betti cohomology groups of the  $\mathbb{C}$ -points of the modular

curves, which are given by evaluation of the cup product on a fundamental class given by the standard orientation of the Riemann surface  $X_1(Np^r)(\mathbb{C})$ . These cup products induce identifications

$$\begin{aligned} H_{\text{ét}}^1(X_1(Np^r)_{/\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) &\xrightarrow{\sim} H_1(X_1(Np^r)(\mathbb{C}), \mathbb{Z}_p), \\ H_{\text{ét}}^1(Y_1(Np^r)_{/\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) &\xrightarrow{\sim} H_1(X_1(Np^r)(\mathbb{C}), \{\text{cusps}\}, \mathbb{Z}_p) \end{aligned}$$

that take a class  $g$  to the unique homology class  $\gamma$  such that the map  $h \mapsto g \cup h$  agrees with evaluating the cohomology class  $g$  on  $\gamma$ .

Now, any unit  $g$  on  $Y_1(Np^r)_{/\overline{\mathbb{Q}}}$  gives rise via Kummer theory to a similarly denoted class in  $H_{\text{ét}}^1(Y_1(Np^r)_{/\overline{\mathbb{Q}}}, \mathbb{Z}_p(1))$ . The order  $\text{ord}_x g$  of the zero of  $g$  at a cusp  $x$  satisfies

$$\text{ord}_x g = g \cup h_x = \partial_x g, \quad (2.1)$$

where  $h_x \in H_{c, \text{ét}}^1(Y_1(Np^r)_{/\overline{\mathbb{Q}}}, \mathbb{Z}_p(1))$  is the image of  $x$  under the canonical connecting map, and where  $\partial_x g$  is the boundary at  $x$  in  $H_0(\{x\}, \mathbb{Z}_p) \cong \mathbb{Z}_p$  of the relative homology class corresponding to  $g$ . These identities can be verified by comparison with de Rham cohomology:

$$g \cup h_x = \frac{1}{2\pi i} \int_X \frac{dg}{g} \wedge d\eta_x = \frac{1}{2\pi i} \int_{\partial D_x} \frac{dg}{g} = \text{ord}_x g$$

for a smooth function  $\eta_x$  that is 1 on a small closed disk  $D_x$  about  $x$  and 0 outside of a larger one in  $Y_1(Np^r)$ . On the other hand, if  $g$  is sent to the class of  $\gamma$  then

$$g \cup h_x = \int_{\gamma} d\eta_x = \sum_y \partial_y g \cdot \eta_x(y) = \partial_x g,$$

where the sum is taken over all cusps  $y$  of  $X_1(Np^r)$ .

## 2.2 Iwasawa modules

**Definition 2.2.1.** Set  $\mathbb{Z}_{p,N} = \varprojlim_r \mathbb{Z}/Np^r\mathbb{Z}$ , set  $\tilde{\Lambda} = \mathbb{Z}_p[[\mathbb{Z}_{p,N}^\times / \langle -1 \rangle]]$ , and set  $\Delta = (\mathbb{Z}/Np\mathbb{Z})^\times$ .

Note that we have a canonical decomposition  $\mathbb{Z}_{p,N}^\times \cong \Delta \times (1 + p\mathbb{Z}_p)$ .

**Definition 2.2.2.** Set  $\Lambda = \mathbb{Z}_p[[1 + p\mathbb{Z}_p]]$ , let  $\chi$  denote the isomorphism

$$\chi = (1 - p^{-1}) \log: 1 + p\mathbb{Z}_p \xrightarrow{\sim} \mathbb{Z}_p,$$

let  $t \in 1 + p\mathbb{Z}_p$  be such that  $\chi(t) = 1$ , let  $\gamma \in \Lambda$  be the group element defined by  $t$ , and set  $X = \gamma - 1 \in \Lambda$ .

Note that these definitions allow us to consider  $\tilde{\Lambda}$  as the  $\Lambda = \mathbb{Z}_p[[X]]$ -algebra  $\Lambda[\Delta/\langle -1 \rangle]$ .

**Definition 2.2.3.**

- a. Set  $\tilde{\Gamma} = \text{Gal}(\mathbb{Q}(\mu_{Np^\infty})^+/\mathbb{Q})$ .
- b. Let  $\mathbb{Q}_\infty$  denote the cyclotomic  $\mathbb{Z}_p$ -extension of  $\mathbb{Q}$ , and set  $\Gamma = \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ .

We have an isomorphism  $\tilde{\Gamma} \xrightarrow{\sim} \mathbb{Z}_{p,N}^\times/\langle -1 \rangle$  given by the cyclotomic character, which we use to identify  $\tilde{\Lambda}$  with  $\mathbb{Z}_p[[\tilde{\Gamma}]]$ . We similarly identify  $\Lambda$  with  $\mathbb{Z}_p[[\Gamma]]$ . We also use this isomorphism to identify  $\Delta/\langle -1 \rangle$  with a subgroup (and quotient) of  $\tilde{\Gamma}$ .

*Remark 2.2.4.* Note that  $\mathfrak{h}^{\text{ord}}$  is a  $\tilde{\Lambda}$ -algebra on which group elements act as inverses of diamond operators. At times, we may work with  $\tilde{\Lambda}$ -modules with distinct actions of inverse diamond operators and Galois elements. The action that we are considering should be discernable from context.

**Definition 2.2.5.**

- a. Let  $\mathbb{Z}_\infty$  denote the integer ring of  $\mathbb{Q}_\infty$ , and let  $\mathbb{Z}_r$  be the ring of integers of the extension  $\mathbb{Q}_r$  of  $\mathbb{Q}$  of degree  $p^{r-1}$  in  $\mathbb{Q}_\infty$  for  $r \geq 1$ .
- b. Set  $\mathcal{O} = \mathbb{Z}[\frac{1}{Np}]$ ,  $\mathcal{O}_\infty = \mathbb{Z}_\infty[\frac{1}{Np}]$ , and  $\mathcal{O}_r = \mathbb{Z}_r[\frac{1}{Np}]$ .
- c. Let  $\mathbb{Q}_{p,\infty}$  denote the cyclotomic  $\mathbb{Z}_p$ -extension of  $\mathbb{Q}_p$ , and let  $\mathbb{Q}_{p,r}$  denote the unique degree  $p^{r-1}$  extension of  $\mathbb{Q}_p$  in  $\mathbb{Q}_{p,\infty}$ .

**Definition 2.2.6.** For any algebraic extension  $F$  of  $\mathbb{Q}$ , we consider the set  $S$  of primes dividing  $Np^\infty$ . We let  $G_{F,S}$  denote the Galois group of the maximal  $S$ -ramified extension of  $F$ .

We may view  $\tilde{\Gamma}$  as a quotient of  $G_{\mathbb{Q},S}$ .

**Definition 2.2.7.** For a  $\tilde{\Lambda}$ -module  $M$ , we consider  $M$  as a  $\tilde{\Lambda}[[G_{\mathbb{Q},S}]]$ -module  $M^l$  by letting  $\sigma \in G_{\mathbb{Q},S}$  act by multiplication by the inverse of its image in  $\tilde{\Gamma}$ .

In particular, by taking completed tensor products with  $\Lambda^l$ , we may define Iwasawa cohomology groups. (Under our conventions,  $\sigma \in G_{\mathbb{Q},S}$  acts on  $\Lambda^l$  by multiplication by the inverse of its image in  $\Gamma$ .)

**Definition 2.2.8.** For a pro- $p$  étale  $\mathcal{O}$ -sheaf (or compact  $\mathbb{Z}_p[[G_{\mathbb{Q},S}]]$ -module)  $\mathcal{F}$ , the  $i$ th  $S$ -ramified Iwasawa cohomology group of  $\mathcal{F}$  is

$$H_{\text{Iw}}^i(\mathcal{O}_\infty, \mathcal{F}) = H^i(\mathcal{O}, \Lambda^l \hat{\otimes}_{\mathbb{Z}_p} \mathcal{F}),$$

where we omit the subscript “ét” from our étale (or, really, continuous Galois) cohomology groups.

We will frequently omit the word ‘‘compact’’ (and ‘‘Hausdorff’’) when considering compact (Hausdorff) modules over a completed group ring with coefficients in a compact (Hausdorff)  $\mathbb{Z}_p$ -algebra.

*Remark 2.2.9.*

- a. We similarly have compactly supported Iwasawa cohomology groups and local-at- $p$  Iwasawa cohomology groups

$$H_{c,\text{Iw}}^i(\mathcal{O}_\infty, \mathcal{F}) = H_c^i(\mathcal{O}, \Lambda^t \hat{\otimes}_{\mathbb{Z}_p} \mathcal{F}) \quad \text{and} \quad H_{\text{Iw}}^i(\mathbb{Q}_{p,\infty}, \mathcal{F}) = H^i(\mathbb{Q}_p, \Lambda^t \hat{\otimes}_{\mathbb{Z}_p} \mathcal{F})$$

for pro- $p$  étale  $\mathcal{O}$  and  $\mathbb{Q}_p$ -sheaves  $\mathcal{F}$ , respectively. We will also consider Iwasawa cohomology for  $\mathcal{O}_\infty[\mu_{Np}]$ , defined using  $\tilde{\Lambda}$  in place of  $\Lambda$ .

- b. Via Shapiro’s lemma, we may make the identification

$$H_{\text{Iw}}^i(\mathcal{O}_\infty, \mathcal{F}) \cong \varprojlim_r H^i(\mathcal{O}_r, \mathcal{F})$$

with the inverse limit taken with respect to corestriction maps, and similarly for the other types of Iwasawa cohomology groups, where in the local setting, the isomorphism is with a product of inverse limits of cohomology groups over primes over  $p$ .

## 2.3 Local actions at $p$

**Definition 2.3.1.** We fix an even  $p$ -adic Dirichlet character  $\theta: \Delta \rightarrow \overline{\mathbb{Q}_p}^\times$ .

**Definition 2.3.2.** For any  $\overline{\mathbb{Q}_p}^\times$ -valued character  $\alpha$  of a group, let  $R_\alpha$  denote the  $\mathbb{Z}_p$ -algebra generated by the values of  $\alpha$ .

We consider  $R_\theta$  as a quotient of  $\mathbb{Z}_p[\Delta]$  via the  $\mathbb{Z}_p$ -linear map to  $R_\theta$  induced by  $\theta$ .

**Definition 2.3.3.** The  $\theta$ -part  $M_\theta$  of a  $\mathbb{Z}_p[\Delta]$ -module  $M$  is the  $R_\theta$ -module

$$M_\theta = M \otimes_{\mathbb{Z}_p[\Delta]} R_\theta.$$

*Remark 2.3.4.* Given a  $\tilde{\Lambda}$ -module  $M$ , we view  $M_\theta$  as a module over the complete local ring  $\Lambda_\theta := R_\theta[[\Gamma]] = R_\theta[[X]]$ . We will most typically think of  $\Lambda_\theta$  as the  $\theta$ -part of the algebra of inverse diamond operators, whereas  $\Lambda$  will often (but not as consistently) be viewed as an algebra of Galois elements.

**Definition 2.3.5.**

- a. Let  $\mathcal{T}_{\text{quo}}^{\text{ord}}$  (resp.,  $\tilde{\mathcal{T}}_{\text{quo}}^{\text{ord}}$ ) denote the maximal unramified  $\mathfrak{H}_{\theta}[[G_{\mathbb{Q}_p}]]$ -quotient of  $\mathcal{T}_{\theta}^{\text{ord}}$  (resp.,  $\tilde{\mathcal{T}}_{\theta}^{\text{ord}}$ ).
- b. Let  $\mathcal{T}_{\text{sub}}^{\text{ord}}$  denote the kernel of the quotient map  $\mathcal{T}_{\theta}^{\text{ord}} \rightarrow \mathcal{T}_{\text{quo}}^{\text{ord}}$ , which is also the kernel of  $\tilde{\mathcal{T}}_{\theta}^{\text{ord}} \rightarrow \tilde{\mathcal{T}}_{\text{quo}}^{\text{ord}}$ .

Ohta [Oh1, Section 4] constructed a perfect ‘‘twisted Poincaré duality’’ pairing

$$\langle \cdot, \cdot \rangle: \mathcal{T}_{\theta}^{\text{ord}} \times \mathcal{T}_{\theta}^{\text{ord}} \rightarrow \Lambda_{\theta}^l(1) \quad (2.2)$$

of  $\Lambda_{\theta}[[G_{\mathbb{Q}_p}]]$ -modules for which  $(Tx, y) = (x, Ty)$  for all  $x, y \in \mathcal{T}_{\theta}$  and  $T \in \mathfrak{h}_{\theta}^{\text{ord}}$ . This is compatible with an analogously defined pairing

$$\langle \cdot, \cdot \rangle: \tilde{\mathcal{T}}_{\theta}^{\text{ord}} \times \tilde{\mathcal{T}}_{c, \theta}^{\text{ord}} \rightarrow \Lambda_{\theta}^l(1) \quad (2.3)$$

of Ohta [Oh2, Theorem 1.3.3] with the same properties, but taking  $T \in \mathfrak{H}_{\theta}^{\text{ord}}$ .

*Remark 2.3.6.* The submodule  $\mathcal{T}_{\text{sub}}^{\text{ord}}$  is isotropic with respect to Ohta’s pairing, yielding a perfect  $\Lambda_{\theta}$ -duality between the  $\mathfrak{h}_{\theta}^{\text{ord}}[[G_{\mathbb{Q}_p}]]$ -modules  $\mathcal{T}_{\text{sub}}^{\text{ord}}$  and  $\mathcal{T}_{\text{quo}}^{\text{ord}}$  [Oh1, Theorem 4.3.1]. As a consequence,  $\mathcal{T}_{\text{sub}}^{\text{ord}}$  is isomorphic to  $\mathfrak{h}_{\theta}^l(1)$  as an  $\mathfrak{h}_{\theta}^{\text{ord}}[[G_{\mathbb{Q}_p^{\text{ur}}}]$ -module, where  $\mathbb{Q}_p^{\text{ur}}$  denotes the maximal unramified extension of  $\mathbb{Q}_p$  [FK, 1.7.13].

**Definition 2.3.7.** For a compact unramified  $\mathfrak{R}[[G_{\mathbb{Q}_p}]]$ -module  $U$  with  $\mathfrak{R}$  a compact  $\mathbb{Z}_p$ -algebra, we define

$$D(U) = (U \hat{\otimes}_{\mathbb{Z}_p} W)^{\text{Fr}_p=1},$$

i.e., the fixed part of the completed tensor product for the diagonal action of the Frobenius  $\text{Fr}_p$ , where  $W$  is the completion of the valuation ring of  $\mathbb{Q}_p^{\text{ur}}$ .

*Remark 2.3.8.* In the notation of Definition 2.3.7, the following hold.

- a. There is a (noncanonical) natural isomorphism between the forgetful functor from compact unramified  $\mathfrak{R}[[G_{\mathbb{Q}_p}]]$ -modules to compact  $\mathfrak{R}$ -modules and  $D$  and under which each  $U \rightarrow D(U)$  is an isomorphism [FK, 1.7.6].
- b. Endowing  $D(U)$  for each  $U$  with the additional action of  $\varphi = 1 \otimes \text{Fr}_p$ , any choice of natural isomorphism as above induces canonical isomorphisms

$$U/(1 - \text{Fr}_p)U \xrightarrow{\sim} D(U)/(1 - \varphi)D(U).$$

The following  $\Lambda$ -adic Eichler-Shimura isomorphisms can be found in [FK, 1.7.9] and extend work of Ohta from [Oh1].



**Theorem 2.3.9** (Ohta, Fukaya-Kato). *We have canonical isomorphisms*

$$D(\mathcal{T}_{\text{quo}}^{\text{ord}}) \cong \mathfrak{S}_{\theta}^{\text{ord}} \quad \text{and} \quad D(\tilde{\mathcal{T}}_{\text{quo}}^{\text{ord}}) \cong \mathfrak{M}_{\theta}^{\text{ord}}$$

of  $\mathfrak{H}_{\theta}^{\text{ord}}$ -modules.

*Remark 2.3.10.* A well-known result of Hida theory (see Ohta [Oh1, Theorem 1.4.3]) states that  $\mathcal{T}_{\theta}^{\text{ord}}$  and  $\tilde{\mathcal{T}}_{\theta}^{\text{ord}}$  are  $\Lambda_{\theta}$ -free of finite rank.

## 2.4 Eisenstein parts and quotients

For an  $\mathfrak{H}^{\text{ord}}$ -module  $M$ , we let  $M_{\mathfrak{m}}$  denote its Eisenstein part: the product of its localizations at the maximal ideals containing  $T_{\ell} - 1 - \ell\langle\ell\rangle$  for primes  $\ell \nmid Np$  and  $U_{\ell} - 1$  for primes  $\ell \mid Np$ .

**Definition 2.4.1.**

- a. We define the cuspidal Hecke algebra  $\mathfrak{h}$  as the Eisenstein part  $\mathfrak{h}_{\mathfrak{m}}^{\text{ord}}$  of Hida's ordinary cuspidal Hecke algebra  $\mathfrak{h}^{\text{ord}}$ .
- b. The Eisenstein ideal  $I$  of  $\mathfrak{h}$  is the ideal generated by  $T_{\ell} - 1 - \ell\langle\ell\rangle$  for primes  $\ell \nmid Np$  and  $U_{\ell} - 1$  for primes  $\ell \mid Np$  in  $\mathfrak{h}_{\mathfrak{m}}$ .

We also set  $\mathfrak{H} = \mathfrak{H}_{\mathfrak{m}}^{\text{ord}}$  and in general use the following notational convention.

**Definition 2.4.2.** For an  $\mathfrak{H}^{\text{ord}}$ -module denoted  $M^{\text{ord}}$ , we set  $M = M_{\mathfrak{m}}^{\text{ord}}$ .

By applying this convention, we obtain  $\mathfrak{H}$ -modules  $\mathfrak{S}$ ,  $\mathfrak{M}$ ,  $\mathcal{I}$ ,  $\mathcal{M}$ ,  $\mathcal{T}$ ,  $\tilde{\mathcal{T}}$ ,  $\mathcal{I}_{\text{quo}}$ ,  $\tilde{\mathcal{I}}_{\text{quo}}$ , and  $\mathcal{I}_{\text{sub}}$ . (Note that  $\mathcal{I}_{\text{sub}}$  and  $\mathcal{I}_{\text{quo}}$  are a submodule and a quotient of  $\mathcal{T}_{\theta}$ , rather than just  $\mathcal{T}$ .) It is only these Eisenstein parts that will be of use to us in the rest of the paper, so we focus solely on them, eschewing greater generality, but obtaining somewhat finer results in the later consideration of zeta elements.

We make the following assumptions on our even character  $\theta$ .

**Hypothesis 2.4.3.** *We suppose that the following conditions on  $\theta$  hold:*

- a.  $p$  divides the generalized Bernoulli number  $B_{2, \theta^{-1}}$ .
- b.  $\theta$  has conductor  $N$  or  $Np$ ,
- c.  $\theta \neq 1, \omega^2$  (if  $N = 1$ ),
- d. either  $\theta \omega^{-1}|_{(\mathbb{Z}/p\mathbb{Z})^{\times}} \neq 1$  or  $\theta|_{(\mathbb{Z}/N\mathbb{Z})^{\times}(p)} \neq 1$ ,

*Remark 2.4.4.* Hypothesis 2.4.3a tells us that  $\mathfrak{h}_\theta \neq 0$ .

Using Hypothesis 2.4.3d, we have the following exactly as in the work of Ohta [Oh2, §3.4] (cf. [FK, 6.3.12]).

**Lemma 2.4.5.** *The exact sequence*

$$0 \rightarrow \mathcal{T}_{\text{sub}} \rightarrow \mathcal{T}_\theta \rightarrow \mathcal{T}_{\text{quo}} \rightarrow 0$$

*is canonically split as a sequence of  $\mathfrak{h}_\theta$ -modules.*

We consider the following power series corresponding to the Kubota-Leopoldt  $p$ -adic  $L$ -function of interest.

**Definition 2.4.6.** Let  $\xi = \xi_\theta \in \Lambda_\theta$  be the element characterized by the property that

$$\xi_\theta(t^s - 1) = L_p(\omega^2 \theta^{-1}, s - 1)$$

for all  $s \in \mathbb{Z}_p$ .

*Remark 2.4.7.* The Mazur-Wiles proof of the main conjecture implies that  $(\mathfrak{h}/I)_\theta \cong \Lambda_\theta/\xi_\theta$  (see Ohta [Oh2, Corollary A.2.4]).

**Definition 2.4.8.** We let  $T = \mathcal{T}_\theta/I\mathcal{T}_\theta$ .

We recall the following from [FK, Section 6.3] (cf. [Sh1, Corollary 4.9]).

**Proposition 2.4.9.** *The reduced lattice  $T$  has a  $(\mathfrak{h}/I)_\theta[[G_{\mathbb{Q},s}]]$ -quotient  $Q$  canonically isomorphic to  $(\mathfrak{h}/I)_\theta^l(1)$ .*

*Proof.* Consider the Manin-Drinfeld modification of the inverse limit of the first homology groups of  $X_1(Np^r)$  relative to the cusps, which is isomorphic to  $\tilde{\mathcal{T}} \otimes_{\mathfrak{h}} \mathfrak{h}$  by [Sh1, Lemma 4.1]. Its quotient by  $\mathcal{T}$  is isomorphic to  $\mathfrak{h}/I$ , generated by the image  $e_\infty$  of the compatible sequence of relative homology classes  $\{0 \rightarrow \infty\}_r$  of the geodesic paths from 0 to  $\infty$  in the upper half-plane [Sh1, Lemma 4.8]. The  $\Lambda_\theta$ -module  $\tilde{\mathcal{T}} \otimes_{\mathfrak{h}} \mathfrak{h}$  is free, as it has no  $X$ -torsion and its quotient by  $X$  is  $R_\theta$ -free as the Manin-Drinfeld modification of the Eisenstein part of the relative homology of  $X_1(Np)$  (cf. [FK, (6.2.9)]). By Remarks 2.3.10 and 2.4.7, we then see that  $\xi e_\infty$  must be an element of a  $\Lambda_\theta$ -basis of  $\mathcal{T}_\theta$  (cf. [FK, (6.2.10)]). The desired surjection is given by  $y \mapsto \langle \xi e_\infty, y \rangle$  on  $y \in \mathcal{T}_\theta$ , using the nondegeneracy of Ohta's pairing (2.2).  $\square$

*Remark 2.4.10.* We have made a sign change here from our original map and that of [FK, 6.3.18]. That is, we pair with  $\xi e_\infty$  on the left, rather than the right.

We define  $P$  as the kernel of the quotient map  $T \rightarrow Q$ , yielding an exact sequence

$$0 \rightarrow P \rightarrow T \rightarrow Q \rightarrow 0 \quad (2.4)$$

of  $(\mathfrak{h}/I)_\theta[[G_{\mathbb{Q},S}]]$ -modules. We recall the following from the main results of [FK, Section 6.3].

**Proposition 2.4.11.** *The canonical maps  $P \rightarrow \mathcal{T}_{\text{quo}}/I\mathcal{T}_{\text{quo}}$  and  $\mathcal{T}_{\text{sub}}/I\mathcal{T}_{\text{sub}} \rightarrow Q$  are isomorphisms of  $(\mathfrak{h}/I)_\theta[[G_{\mathbb{Q},S}]]$ -modules. Moreover, the action of  $G_{\mathbb{Q},S}$  on  $P$  is trivial, and  $P$  can be identified with the fixed part of  $T$  under any complex conjugation.*

*Proof.* The cokernel of the map  $\pi: \mathcal{T}_{\text{sub}}/I\mathcal{T}_{\text{sub}} \rightarrow Q$  is an  $(\mathfrak{h}/I)_\theta[[G_{\mathbb{Q},S}]]$ -module quotient of  $\mathcal{T}_{\text{quo}}/I\mathcal{T}_{\text{quo}}$ . The  $\Delta_p$ -action on  $\mathcal{T}_{\text{quo}}/I\mathcal{T}_{\text{quo}}$  is trivial, while the  $\Delta_p$ -action on  $Q$  is via  $\omega\theta^{-1}$ , so by Hypothesis 2.4.3d, we have that  $\pi$  is surjective. Moreover,  $\mathcal{T}_{\text{sub}}/I\mathcal{T}_{\text{sub}}$  and  $Q$  are both free of rank one over  $(\mathfrak{h}/I)_\theta$ , so  $\pi$  must also be injective. This forces the other map to be an isomorphism as well.

Next, let us briefly outline the argument of Kurihara and Harder-Pink yielding the triviality of the action on  $P$ , as in [FK, 6.3.15]. By Lemma 2.4.5, we have a direct sum decomposition  $T = P \oplus Q$  as  $(\mathfrak{h}/I)_\theta$ -modules, with  $P$  being  $G_{\mathbb{Q},S}$ -stable. The character defining the determinant of the action of  $G_{\mathbb{Q},S}$  on the modular representation in which  $\mathcal{T}_\theta$  is a lattice reduces exactly to the character defining the action on  $Q$ . Consequently,  $G_{\mathbb{Q},S}$  must act trivially on  $P$ . Since complex conjugation then acts trivially on  $P$  and as  $-1$  on the quotient  $Q$ , we have the final claim.  $\square$

**Corollary 2.4.12.** *The maps  $\mathcal{T}_\theta^+ \rightarrow \mathcal{T}_{\text{quo}}$  and  $\tilde{\mathcal{T}}_\theta^+ \rightarrow \tilde{\mathcal{T}}_{\text{quo}}$  are isomorphisms.*

*Proof.* The maps  $\mathcal{T}_\theta/\mathcal{T}_\theta^+ \rightarrow \tilde{\mathcal{T}}_\theta/\tilde{\mathcal{T}}_\theta^+$  and  $\mathcal{T}_{\text{sub}} \rightarrow \tilde{\mathcal{T}}_{\text{sub}}$  are isomorphisms, so it suffices to show that  $\mathcal{T}_{\text{sub}} \rightarrow \mathcal{T}_\theta/\mathcal{T}_\theta^+$  is an isomorphism. We know that it is surjective by Proposition 2.4.11 and Nakayama's lemma. But  $\mathcal{T}_{\text{sub}}$  is a free  $\mathfrak{h}_\theta$ -module of rank 1, and  $\mathcal{T}_\theta/\mathcal{T}_\theta^+$  is an  $\mathfrak{h}_\theta$ -module of rank 1, so the surjectivity forces the map to be an isomorphism.  $\square$

As in [FK, 6.3.4], we see that our sequence (2.4) is uniquely locally split.

**Proposition 2.4.13.** *The sequence (2.4) is uniquely split as a sequence of  $(\mathfrak{h}/I)_\theta[[G_{\mathbb{Q},\ell}]]$ -modules for every  $\ell \mid Np$ .*

*Proof.* For  $\ell = p$ , this is a direct consequence of Proposition 2.4.11. For  $\ell \mid N$ , this follows from Hypothesis 2.4.3b and facts that the decomposition group  $\Delta_\ell$  at  $\ell$  in  $\Delta$  acts trivially on  $P$  and via  $\omega\theta^{-1}$  on  $Q$ .  $\square$

We also have the following results on  $P$ .

*Remarks 2.4.14.*

- a. The  $G_{\mathbb{Q},S}$ -action on  $P$  is trivial, and we have a canonical isomorphism  $P \cong \mathfrak{S}_\theta / I\mathfrak{S}_\theta$  of  $\mathfrak{h}_\theta$ -modules. For this, note that  $U_p$  acts as an arithmetic Frobenius on  $\mathcal{T}_{\text{quo}}$  by [FK, 1.8.1] and that  $D(\mathcal{T}_{\text{quo}}) \cong \mathfrak{S}_\theta$ , and apply Proposition 2.4.11 and Remark 2.3.8(b).
- b. The  $p$ -adic  $L$ -function  $\xi$  divides the  $\Lambda_\theta$ -characteristic ideal of  $P$  (for the action of inverse diamond operators) by an argument of Mazur-Wiles and Ohta (see [FK, 7.1.3]).

Putting these isomorphisms together with Remark 2.3.8a and Proposition 2.3.9, we have isomorphisms  $\mathfrak{S}_\theta \cong \mathcal{T}_\theta^+ \cong \mathcal{S}_\theta$  and  $\mathfrak{M}_\theta \cong \tilde{\mathcal{T}}_\theta^+ \cong \mathcal{M}_\theta$  on Eisenstein components. Note that the first of each of these pairs of isomorphisms is noncanonical, only becoming canonical upon reduction modulo  $U_p - 1$ , but we can and do fix compatible choices.

### 3 Cohomological study

In this section, we first introduce known results on the cohomology of the reduced lattice that is the quotient  $T$  of  $\mathcal{T}_\theta$  by the Eisenstein ideal. We recall the work of Fukaya and Kato [FK] in which the derivative  $\xi'$  of a Kubota-Leopoldt  $p$ -adic  $L$ -function  $\xi$  appears in the study of certain connecting homomorphisms in the cohomology of subquotients of  $T(1)$ . We then perform an analogous study, replacing  $T$  by a certain ‘‘intermediate’’ quotient  $T^\dagger$  of  $\Lambda^1 \hat{\otimes}_{\mathbb{Z}_p} T$ , and we show that in this setting the role of  $\xi'$  is played more simply by 1.

#### 3.1 Cohomology of the reduced lattice

**Definition 3.1.1.** We set  $Y = H_{\text{Iw}}^2(\mathcal{O}_\infty[\mu_{Np}], \mathbb{Z}_p(2))_\theta$  and consider it as a  $\Lambda_\theta$ -module for the action of inverse diamond operators.

*Remark 3.1.2.* Let  $Y'$  denote the  $\theta$ -eigenspace of the Tate twist of the minus part of the unramified Iwasawa module over  $\mathbb{Q}(\mu_{Np^\infty})$ . Then the canonical maps

$$Y' \rightarrow H_{\text{Iw}}^2(\mathbb{Z}_\infty[\frac{1}{p}], \mu_{Np}, \mathbb{Z}_p(2))_\theta \hookrightarrow Y$$

are isomorphisms by our hypotheses on  $\theta$ . In particular, the characteristic ideal of  $Y$  is generated by  $\xi$  by the Iwasawa main conjecture.

*Remark 3.1.3.* It follows from Shapiro’s lemma that

$$H^i(\mathcal{O}, \Lambda_\theta^1(2)) \cong H_{\text{Iw}}^i(\mathcal{O}_\infty[\mu_{Np}], \mathbb{Z}_p(2))_\theta$$

for  $i \in \mathbb{Z}$ . In particular, we may identify  $Y$  with  $H^2(\mathcal{O}, \Lambda_\theta^1(2))$ .

We recall the following [FK, 9.1.4].

**Lemma 3.1.4** (Fukaya-Kato). *The cohomology groups  $H^i(\mathcal{O}, Q(1))$  are zero for  $i \notin \{1, 2\}$  and are isomorphic to  $Y$  otherwise, More precisely, the connecting map in the long exact sequence attached to*

$$0 \rightarrow \Lambda_\theta^1(2) \xrightarrow{\xi} \Lambda_\theta^1(2) \rightarrow Q(1) \rightarrow 0$$

*induces an isomorphism  $H^1(\mathcal{O}, Q(1)) \xrightarrow{\sim} Y$  and the quotient map in said sequence induces an isomorphism  $Y \xrightarrow{\sim} H^2(\mathcal{O}, Q(1))$ .*

*Proof.* The group  $H^1(\mathcal{O}_\infty, \Lambda_\theta^1(2))$  vanishes since it is isomorphic to the Tate twist of the group of norm compatible systems of  $p$ -completions of  $p$ -units in the cyclotomic  $\mathbb{Z}_p$ -extension of  $\mathbb{Q}(\mu_{Np})$ , its  $\theta$ -eigenspace is zero since  $\theta$  is even, not equal to  $\omega^2$ , and Hypothesis 2.4.3d holds. Since  $G_{\mathbb{Q}, S}$  has  $p$ -cohomological dimension 2, we have an exact sequence

$$0 \rightarrow H^1(\mathcal{O}, Q(1)) \rightarrow H^2(\mathcal{O}, \Lambda_\theta^1(2)) \xrightarrow{\xi} H^2(\mathcal{O}, \Lambda_\theta^1(2)) \rightarrow H^2(\mathcal{O}, Q(1)) \rightarrow 0$$

in which the middle map is zero by Stickelberger theory (or the main conjecture and the fact that  $Y$  has no  $p$ -torsion).  $\square$

We also note the following simple lemma on the compactly-supported cohomology of  $P$ .

**Lemma 3.1.5.** *The compactly supported cohomology groups  $H_c^i(\mathcal{O}, P(1))$  are zero for  $i \notin \{2, 3\}$  and are isomorphic to  $P$  otherwise. For  $i = 3$ , the isomorphism is given by the invariant map, whereas for  $i = 2$ , we have a canonical isomorphism  $H_c^2(\mathcal{O}, P(1)) \cong \Gamma \hat{\otimes}_{\mathbb{Z}_p} P$  of Poitou-Tate duality that we compose with the map induced by  $-\chi: \Gamma \rightarrow \mathbb{Z}_p$ . Moreover, the natural maps*

$$H_c^i(\mathbb{Z}[\frac{1}{p}], P(1)) \rightarrow H_c^i(\mathcal{O}, P(1))$$

*are isomorphisms.*

*Proof.* This is well-known for  $i = 3$ , since the compactly supported cohomology of  $\mathcal{O}$  has  $p$ -cohomological dimension 3 and  $P$  has trivial Galois action. That is, we have canonical isomorphisms

$$H_c^3(\mathcal{O}, P(1)) \cong H_c^3(\mathcal{O}, \mathbb{Z}_p(1)) \hat{\otimes}_{\mathbb{Z}_p} P \cong H^0(\mathcal{O}, \mathbb{Q}_p/\mathbb{Z}_p)^\vee \hat{\otimes}_{\mathbb{Z}_p} P \cong P.$$

For  $i = 1$ , we similarly have

$$H_c^1(\mathcal{O}, P(1)) \cong H^2(\mathcal{O}, P^\vee)^\vee = 0$$

in that  $H^2(\mathcal{O}, \mathbb{Z}/p\mathbb{Z}) = 0$ . Since the above arguments work for any compact  $\mathbb{Z}_p$ -module  $M$  with trivial  $G_{\mathbb{Q}, S}$ -action, the functors  $M \mapsto H_c^i(\mathcal{O}, M(1))$  are exact for  $i = 2, 3$  and are trivial for all other  $i$ . The maximal pro- $p$ , abelian,  $S$ -ramified extension of  $\mathbb{Q}$  is  $\mathbb{Q}_\infty$  in that no prime dividing  $N$  is 1 modulo  $p$ , so we have

$$H_c^2(\mathcal{O}, P(1)) \cong H_c^2(\mathcal{O}, \mathbb{Z}_p(1)) \hat{\otimes}_{\mathbb{Z}_p} P \cong H^1(\mathcal{O}, \mathbb{Q}_p/\mathbb{Z}_p)^\vee \hat{\otimes}_{\mathbb{Z}_p} P \cong \Gamma \hat{\otimes}_{\mathbb{Z}_p} P.$$

and we apply the isomorphism  $-\chi: \Gamma \rightarrow \mathbb{Z}_p$  to obtain the result. A similar argument gives the analogous statements for  $\mathbb{Z}[\frac{1}{p}]$  and through it the isomorphisms.  $\square$

We can define a cocycle  $b: G_{\mathbb{Q}, S} \rightarrow \text{Hom}_{\mathfrak{h}}(Q, P)$  using the exact sequence (2.4) by

$$b(\sigma)(q) = \sigma(\widetilde{\sigma^{-1}q}) - \tilde{q}$$

for  $q \in Q$ , letting  $\tilde{x}$  denote the image of  $x$  under a fixed  $\mathfrak{h}_\theta$ -module splitting  $Q \rightarrow T$ . Then  $b$  restricts to an everywhere unramified homomorphism on the absolute Galois group of  $\mathbb{Q}(\mu_{Np^\infty})$  by Proposition 2.4.13, which we can view as having domain  $Y$  by Remark 3.1.2. Through the isomorphism of Proposition 2.4.9, we have moreover a canonical isomorphism  $\text{Hom}_{\mathfrak{h}}(Q, P) \cong P$  of  $\Lambda_\theta$ -modules. The result is the desired map  $\Upsilon$  (see [Sh1, Section 4.4], though note that we have not multiplied by any additional unit here).

**Definition 3.1.6.** Let  $\Upsilon: Y \rightarrow P$  denote the homomorphism of  $\Lambda_\theta$ -modules induced by  $b$  and Proposition 2.4.9.

We also have a map in the other direction that takes a trace-compatible system of Manin symbols to a corestriction compatible system of cup products of cyclotomic units.

**Definition 3.1.7.** Let  $\varpi: \mathcal{S}_\theta \rightarrow Y$  denote the map constructed in [Sh1, Proposition 5.7], with reference to [FK, 5.2.3], where the latter is shown to factor through  $P$ .

We also use  $\varpi$  to denote the induced map  $\varpi: P \rightarrow Y$ .

*Remark 3.1.8.* We recall that  $\varpi$  is the restriction of the inverse limit under trace and corestriction of maps

$$\varpi_r: H_1(X_1(Np^r), C_1^0(Np^r), \mathbb{Z}_p)^+ \rightarrow H_{\text{ét}}^2(\mathbb{Z}[\mu_{Np^r}, \frac{1}{Np}], \mathbb{Z}_p(2))^+, [u: v]_r \mapsto (1 - \zeta_{Np^r}^u, 1 - \zeta_{Np^r}^v)_r, \quad (3.1)$$

for  $r \geq 1$ , where  $u, v \in \mathbb{Z}/Np^r\mathbb{Z} - \{0\}$  with  $(u, v) = (1)$ . We briefly define the symbols that appear.

On the right hand side of (3.1), the symbol  $(\ , \ )_r$  denotes the pairing on cyclotomic  $Np$ -units induced by the cup product

$$H_{\text{ét}}^1(\mathbb{Z}[\mu_{Np^r}, \frac{1}{Np}], \mathbb{Z}_p(1)) \times H_{\text{ét}}^1(\mathbb{Z}[\mu_{Np^r}, \frac{1}{Np}], \mathbb{Z}_p(1)) \xrightarrow{\cup} H_{\text{ét}}^2(\mathbb{Z}[\mu_{Np^r}, \frac{1}{Np}], \mathbb{Z}_p(2))$$

and Kummer theory, noting that the image of the pairing lands in the plus part of  $H^2$ .

On the left hand side of (3.1), we have

$$[u : v]_r = (w_r \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \{0 \rightarrow \infty\}_r)^+,$$

where  $w_r$  is the Atkin-Lehner involution of level  $Np^r$  and the matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$  has bottom row  $(u, v) \bmod Np^r$ . (Note that  $w_r \{0 \rightarrow \infty\}_r = \{\infty \rightarrow 0\}_r$ .) We project the resulting element to the plus part after the operations, denoting this with  $(\ )^+$ . Since  $u, v \neq 0$ , the symbol  $[u : v]_r$  lies in  $H_1(X_1(Np^r), C_1^0(Np^r), \mathbb{Z}_p)^+$ , where  $C_1^0(Np^r)$  denotes the cusps not over  $0 \in X_0(Np^r)$ .

We recall the conjecture of [Sh1].

**Conjecture 3.1.9.** *The maps  $\varpi : P \rightarrow Y$  and  $\Upsilon : Y \rightarrow P$  are inverse maps.*

Actually, Conjecture 3.1.9 was originally conjectured by the author up to a canonical unit. There were indications that this unit might be 1 (if sign conventions were correct), but while the author advertised this suspicion rather widely and included it in preprint versions of the paper, he opted not to conjecture it in the final published version. It was the work of Fukaya and Kato in [FK] that finally made it clear that the unit should indeed be 1, not least because one would expect that the hypotheses under which they can prove it should hold without exception. Nevertheless, one does not actually know how to prove that their hypotheses always hold. Indeed, this paper is motivated by a desire to explore where the difficulty lies in removing them.

*Remark 3.1.10.* Hida theory tells us that the  $\Lambda_\theta$ -characteristic ideal of  $P$  is divisible by  $(\xi_\theta)$ , and the main conjecture of Iwasawa theory as proven by Mazur-Wiles tells us that the  $\Lambda_\theta$ -characteristic ideal of  $Y$  is equal to  $(\xi_\theta)$ . As  $Y$  is well-known to be  $p$ -torsion free (i.e., by results of Iwasawa and Ferrero-Washington), Conjecture 3.1.9 is reduced to showing that  $\Upsilon \circ \varpi = 1$  on  $P$ .

Consider the complex

$$C_f(\mathcal{O}, T(1)) = \text{Cone} \left( C(\mathcal{O}, T(1)) \rightarrow \bigoplus_{\ell|Np} C(\mathbb{Q}_\ell, P(1)) \right) [-1],$$

where “ $C$ ” here is used to denote the standard cochain complexes and the map in the cone uses the local splitting  $T \rightarrow P$ . We have an exact sequence of complexes

$$0 \rightarrow C_c(\mathcal{O}, P(1)) \rightarrow C_f(\mathcal{O}, T(1)) \rightarrow C(\mathcal{O}, Q(1)) \rightarrow 0,$$

where  $C_c$  is the complex defining compactly supported cohomology, and which has connecting homomorphisms

$$H^i(\mathcal{O}, Q(1)) \rightarrow H_c^{i+1}(\mathcal{O}, P(1))$$

for  $i \geq 0$ . For  $i = 1$ , let us denote this connecting homomorphism by  $\Theta$ . The connecting homomorphism for  $i = 2$  can be identified with  $\Upsilon$ : see [FK, 9.4.3], though note that we obtain that they obtain the opposite sign. One can simply take this as the definition of  $\Upsilon$  for the purposes of this article. Nevertheless, we give a fairly detailed sketch of the proof using the results of [Sh2], as it is by now an old result due independently to the author.

**Lemma 3.1.11.** *Under the identifications of Lemmas 3.1.4 and 3.1.5, the connecting homomorphism*

$$H^2(\mathcal{O}, Q(1)) \rightarrow H_c^3(\mathcal{O}, P(1))$$

is  $\Upsilon: Y \rightarrow P$ .

*Proof.* We consider a diagram

$$\begin{array}{ccc} H_{\text{Iw}}^2(\mathcal{O}_\infty[\mu_{Np}], \mathbb{Z}_p(2))_\theta & \longrightarrow & H_{c, \text{Iw}}^3(\mathcal{O}_\infty[\mu_{Np}], P(1)) \\ \downarrow \wr & & \downarrow \wr \\ H^2(\mathcal{O}, Q(1)) & \longrightarrow & H_c^3(\mathcal{O}, P(1)), \end{array}$$

where the connecting homomorphism that is the lower map is given by left cup product with  $b: G_{\mathbb{Q}, S} \rightarrow \text{Hom}_{\mathfrak{f}}(Q, P)$  by [Sh2, Proposition 2.3.3]. The left vertical map employs the surjection  $\Lambda_\theta^1(1) \rightarrow Q$  determined by Proposition 2.4.9, and the right vertical map uses the quotient map  $\tilde{\Lambda}^1 \rightarrow \mathbb{Z}_p$ , which is to say it becomes corestriction via Shapiro’s lemma. The diagram is then commutative taking the upper horizontal map to be given by left cup product with the cocycle  $G_{\mathbb{Q}(\mu_{Np^\infty}), S} \rightarrow P$  given by following the restriction of  $b$  with evaluation at the canonical generator of  $Q$ . Recall that this cocycle is a homomorphism that by definition factors through  $\Upsilon: Y \rightarrow P$ . That the upper horizontal map then agrees with  $\Upsilon$  via the identifications of the groups with  $Y$  and  $P$  is seen by noting that it is Pontryagin dual via Poitou-Tate duality to the Pontryagin dual of  $\Upsilon$ , via an argument mimicking the proof of [Sh2, Proposition 3.1.3] (noting Proposition 2.4.3 therein, which in particular implies that the signs agree).  $\square$



*Remark 3.1.12.* The connecting map  $H^2(\mathcal{O}, Q(1)) \rightarrow H_c^3(\mathcal{O}, P(1))$  that we use is the negative of the corresponding map in [FK], since the identification of  $Q$  with  $(\mathfrak{h}/I)_\theta^1(1)$  of Proposition 2.4.9, and hence of  $Y$  with  $H^2(\mathcal{O}, Q(1))$ , is of opposite sign to that of [FK, 6.4.3].

**Definition 3.1.13.** For a  $\mathbb{Z}_p[[G_{\mathbb{Q},S}]]$ -module  $M$ , let  $\partial_M$  denote a connecting homomorphism in a long exact sequence in cohomology attached to the Tate twist of the exact sequence

$$0 \rightarrow M \xrightarrow{X} \Lambda^1/(X^2) \hat{\otimes}_{\mathbb{Z}_p} M \xrightarrow{\text{mod } X} M \rightarrow 0. \quad (3.2)$$

*Remark 3.1.14.* The maps  $\partial_M$  for any  $\mathbb{Z}_p[[G_{\mathbb{Q},S}]]$ -module  $M$  agree with left cup product by the cocycle  $-\chi$  defining the extension class (3.2) (cf. [Sh2, Proposition 2.3.3]). As pointed out in [FK, 9.3.4], the sign in  $-\chi$  occurs as  $G_{\mathbb{Q}}$  acts on  $\Lambda^1$  through left multiplication by the inverse of its quotient map to  $\Gamma \subset \Lambda$ .

**Lemma 3.1.15.** *Let  $M$  be a compact or discrete  $\mathbb{Z}_p[[G_{\mathbb{Q},S}]]$ -module. Then the diagram*

$$\begin{array}{ccc} \bigoplus_{\ell|Np} H^1(\mathbb{Q}_\ell, M(1)) & \xrightarrow{\partial_M} & \bigoplus_{\ell|Np} H^2(\mathbb{Q}_\ell, M(1)) \\ \downarrow & & \downarrow \\ H_c^2(\mathcal{O}, M(1)) & \xrightarrow{\partial_M} & H_c^3(\mathcal{O}, M(1)) \end{array}$$

*anticommutes.*

*Proof.* Recall that

$$C_c(\mathcal{O}, M(1)) = \ker \left( C(\mathcal{O}, M(1)) \xrightarrow{\text{res}} \bigoplus_{\ell|Np} C(\mathbb{Q}_\ell, M(1)) \right) [-1]$$

which is to say that

$$C_c^i(\mathcal{O}, M(1)) = \left( \bigoplus_{\ell|Np} C^{i-1}(\mathbb{Q}_\ell, M(1)) \right) \oplus C^i(\mathcal{O}, M(1)),$$

with the differential taking  $(x, y)$  to  $(-d^{i-1}(x) - \text{res}(y), d^i(y))$ . The composition

$$H^1(\mathbb{Q}_\ell, M(1)) \xrightarrow{\partial_M} H^2(\mathbb{Q}_\ell, M(1)) \rightarrow H_c^3(\mathcal{O}, M(1))$$

takes a class  $\phi$  to the image of the compactly-supported cocycle  $(\partial_M(\phi), 0)$ , whereas the composition

$$H^1(\mathbb{Q}_\ell, M(1)) \rightarrow H_c^2(\mathcal{O}, M(1)) \xrightarrow{\partial_M} H_c^3(\mathcal{O}, M(1))$$

takes  $\phi$  to  $\partial_M(\phi, 0) = (-\partial_M(\phi), 0)$  in that the differential used to compute the connecting homomorphism restricts to the negative of the local differential.  $\square$

We also have the following lemma.

**Lemma 3.1.16.** *The connecting homomorphism  $\partial_P: H_c^2(\mathcal{O}, P(1)) \rightarrow H_c^3(\mathcal{O}, P(1))$  is identified with the identity map on  $P$  via the isomorphisms of Lemma 3.1.5.*

*Proof.* As noted in Remark 3.1.14, the connecting map  $\partial_P$  is given by left cup product with  $-\chi \in H^1(\mathcal{O}, \mathbb{Z}_p)$ . By the commutativity (with elements of the even degree cohomology group  $H_c^2(\mathcal{O}, P(1))$ ) and associativity of cup products,  $\partial_P$  is Poitou-Tate dual to the map  $H^0(\mathcal{O}, P^\vee) \rightarrow H^1(\mathcal{O}, P^\vee)$  also given by left cup product with  $-\chi$ . In turn, this dual is identified with the dual of the isomorphism  $\Gamma \otimes_{\mathbb{Z}_p} P \rightarrow P$  induced by  $-\chi: \Gamma \rightarrow \mathbb{Z}_p$ . Finally, this is exactly our prior identification of  $H_c^2(\mathcal{O}, P(1))$  with  $P \cong H_c^3(\mathcal{O}, P(1))$ .  $\square$

The following exercise in Galois cohomology encapsulates a key aspect of the work of Fukaya-Kato [FK, Sections 9.3-9.5]. We omit the proof, as the reader will find its key ideas contained in the refined study that follows (cf. Proposition 3.3.9 for the commutativity of the lefthand square and Lemma 3.3.6 for the middle square on the right).

**Theorem 3.1.17** (Fukaya-Kato). *Let  $\xi' = \xi'_\theta \in \Lambda_\theta$  be such that*

$$\xi'_\theta(t^s - 1) = L'_p(\omega^2 \theta^{-1}, s - 1)$$

for all  $s \in \mathbb{Z}_p$ . Then the diagram

$$\begin{array}{ccccc}
 & & Y & \xrightarrow{\xi'} & Y \\
 & & \wr \uparrow & & \downarrow \wr \\
 H^1(\mathcal{O}, T(1)) & \longrightarrow & H^1(\mathcal{O}, Q(1)) & \xrightarrow{\partial_Q} & H^2(\mathcal{O}, Q(1)) \\
 \downarrow & & \downarrow -\Theta & & \downarrow \Upsilon \\
 \bigoplus_{\ell|N_p} H^1(\mathbb{Q}_\ell, P(1)) & \longrightarrow & H_c^2(\mathcal{O}, P(1)) & \xrightarrow{\partial_P} & H_c^3(\mathcal{O}, P(1)) \\
 & & \wr \uparrow & & \downarrow \wr \\
 & & P & \xrightarrow[\sim]{\text{id}} & P
 \end{array}$$

commutes, where the indicated vertical isomorphisms are those of Lemmas 3.1.4 and 3.1.5, and the leftmost vertical map uses the unique local splittings of Proposition 2.4.13.

In particular, note that  $\Theta$  is identified with  $-\xi' \Upsilon$  as a map  $Y \rightarrow P$ . In this section, we aim to remove the derivative by modifying the diagram.

## 3.2 Intermediate quotients

Let  $\hat{\otimes}_{\mathbb{Z}_p}$  denote the completed tensor product over  $\mathbb{Z}_p$ . We use it consistently even in cases for which the usual tensor product gives the same module (in part, to indicate that our modules carry a compact topology). In the following, when we write  $\Lambda$  (as opposed to  $\Lambda^l$ ), we shall consider it as carrying a trivial  $G_{\mathbb{Q},S}$ -action.

**Definition 3.2.1.** Define

$$w: \Lambda^l \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta^l(1) \xrightarrow{\sim} \Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta^l(1)$$

to be the continuous  $(\Lambda \hat{\otimes}_{\mathbb{Z}_p} R_\theta)[[G_{\mathbb{Q},S}]]$ -module isomorphism satisfying

$$w(\sigma \otimes \tau) = \sigma \tau^{-1} \otimes \tau$$

for  $\sigma, \tau \in \Gamma$ .

Note that multiplication by  $\xi \in \Lambda_\theta$  provides injective endomorphisms of  $\Lambda_\theta$  and  $\Lambda_\theta^l$ .

**Definition 3.2.2.** Define a  $(\Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta)[[G_{\mathbb{Q},S}]]$ -module homomorphism

$$\tilde{\xi}: \Lambda^l \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta^l(1) \rightarrow \Lambda^l \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta^l(1).$$

by  $\tilde{\xi} = w^{-1}(1 \otimes \xi)w$ .

Let  $a_i \in R_\theta$  be such that  $\xi = \sum_{i=0}^{\infty} a_i X^i$ . We then have that  $\tilde{\xi}$  is given by multiplication by the identically denoted element

$$\tilde{\xi} = \sum_{i=0}^{\infty} a_i (\gamma \otimes \gamma - 1)^i \in \Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta.$$

**Definition 3.2.3.** For  $n \geq 0$ , define  $\xi^{(n)}: \Lambda_\theta^l \rightarrow \Lambda_\theta^l$  to be the continuous  $\Lambda_\theta[[G_{\mathbb{Q},S}]]$ -module homomorphism given by multiplication by

$$\xi^{(n)} = (X+1)^n \frac{1}{n!} \frac{d^n \xi}{dX^n} = (X+1)^n \sum_{i=n}^{\infty} a_i \binom{i}{n} X^{i-n} \in \Lambda_\theta.$$

*Remark 3.2.4.* Note that  $\xi^{(1)}$  is  $\xi'$  of Proposition 3.1.17.

We make the identification

$$\Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta \cong R_\theta[[X \otimes 1, 1 \otimes X]]$$

and frequently refer to  $X \otimes 1 \in \Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta$  more simply by  $X$ .

While not used later, the following description of  $\tilde{\xi}$  gives one some insight into its form.

**Proposition 3.2.5.** *We have  $\tilde{\xi} = \sum_{n=0}^{\infty} X^n \otimes \xi^{(n)}$ .*

*Proof.* For  $n \geq 1$ , set

$$\tilde{\xi}_n = X^{-n} \left( \tilde{\xi} - \sum_{i=0}^{n-1} X^i \otimes \xi^{(i)} \right),$$

which we aim to prove lie in  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_{\theta}$ , so  $X^n \tilde{\xi}_n$  tends to zero. It suffices to show that

$$\tilde{\xi}_n = (1 \otimes \gamma)^n \sum_{i=n}^{\infty} a_i \sum_{j=n}^i \binom{j-1}{n-1} (1 \otimes \gamma - 1)^{j-n} (\gamma \otimes \gamma - 1)^{i-j}. \quad (3.3)$$

First, noting the simple identity  $(x-y)^i = (x-y) \sum_{j=1}^i x^{j-1} y^{i-j}$ , we have that

$$\begin{aligned} X \tilde{\xi}_1 &= \sum_{i=0}^{\infty} a_i ((\gamma \otimes \gamma - 1)^i - (1 \otimes \gamma - 1)^i) \\ &= (X \otimes \gamma) \sum_{i=1}^{\infty} a_i \sum_{j=1}^i (1 \otimes \gamma - 1)^{j-1} (\gamma \otimes \gamma - 1)^{i-j}. \end{aligned}$$

Suppose then that (3.3) holds for some  $n \geq 1$ . Since  $X \tilde{\xi}_{n+1} = \tilde{\xi}_n - X^n \otimes \xi^{(n)}$ , we have

$$\begin{aligned} (1 \otimes \gamma)^{-n} \tilde{\xi}_{n+1} &= X^{-1} \sum_{i=n}^{\infty} a_i \sum_{j=n}^i \binom{j-1}{n-1} (1 \otimes \gamma - 1)^{j-n} (\gamma \otimes \gamma - 1)^{i-j} - \sum_{i=n}^{\infty} a_i \binom{i}{n} (1 \otimes \gamma - 1)^{i-n} \\ &= X^{-1} \sum_{i=n+1}^{\infty} a_i \sum_{j=n+1}^i \binom{j-1}{n-1} (1 \otimes \gamma - 1)^{j-n} ((\gamma \otimes \gamma - 1)^{i-j} - (1 \otimes \gamma - 1)^{i-j}) \\ &= (1 \otimes \gamma) \sum_{i=n+1}^{\infty} a_i \sum_{j=n+1}^i \binom{j-1}{n-1} \sum_{k=0}^{i-j-1} (1 \otimes \gamma - 1)^{k+j-n} (\gamma \otimes \gamma - 1)^{i-j-k-1} \\ &= (1 \otimes \gamma) \sum_{i=n+1}^{\infty} a_i \sum_{j=n+1}^i \binom{j-1}{n-1} \sum_{k=j+1}^i (1 \otimes \gamma - 1)^{k-n-1} (\gamma \otimes \gamma - 1)^{i-k} \\ &= (1 \otimes \gamma) \sum_{i=n+1}^{\infty} a_i \sum_{k=n+1}^i \sum_{j=1}^{k-1} \binom{j-1}{n-1} (1 \otimes \gamma - 1)^{k-n-1} (\gamma \otimes \gamma - 1)^{i-k} \\ &= (1 \otimes \gamma) \sum_{i=n+1}^{\infty} a_i \sum_{k=n+1}^i \binom{k-1}{n} (1 \otimes \gamma - 1)^{k-n-1} (\gamma \otimes \gamma - 1)^{i-k}, \end{aligned}$$

completing the induction. □

### 3.3 Refined cohomological study

Let us set

$$T^\dagger = (\Lambda^l \hat{\otimes}_{\mathbb{Z}_p} T) / \tilde{\xi} (\Lambda^l \hat{\otimes}_{\mathbb{Z}_p} T),$$

and similarly for  $P$  and  $Q$ . We shall give this dagger notation a more general definition in Section 4. We first consider  $Q^\dagger$ .

**Definition 3.3.1.** Given a  $\Lambda_\theta$ -module  $M$  for which we consider  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} M$  as a  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta$ -module whereby  $f \in \Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta$  acts by multiplication by  $w(f)$ , we let  $\xi$  denote the  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta$ -module endomorphism of  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} M$  induced by the action of  $1 \otimes \xi$ .

*Remark 3.3.2.* Viewed as an element of the ring  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta$ , we have  $\xi = (\iota \otimes 1)(\tilde{\xi}) \in \Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta$ .

**Proposition 3.3.3.** *We have isomorphisms*

$$H^i(\mathcal{O}, Q^\dagger(1)) \cong (\Lambda \hat{\otimes}_{\mathbb{Z}_p} H^i(\mathcal{O}, Q(1))) / \xi$$

of  $(\Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta)$ -modules for all  $i$ , where  $f \in \Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta$  acts as usual on the left and as  $w(f)$  on  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} H^i(\mathcal{O}, Q(1))$  on the right.

*Proof.* We have a commutative diagram

$$\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta^l(1) & \xrightarrow{\xi} & \Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta^l(1) & \longrightarrow & \Lambda^l \hat{\otimes}_{\mathbb{Z}_p} Q & \longrightarrow & 0 \\
& & \downarrow 1 \otimes \xi & & \downarrow 1 \otimes \xi & & \downarrow \tilde{\xi} & & \\
0 & \longrightarrow & \Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta^l(1) & \xrightarrow{\xi} & \Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta^l(1) & \longrightarrow & \Lambda^l \hat{\otimes}_{\mathbb{Z}_p} Q & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Lambda \hat{\otimes}_{\mathbb{Z}_p} Q & \xrightarrow{\xi} & \Lambda \hat{\otimes}_{\mathbb{Z}_p} Q & \longrightarrow & Q^\dagger & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & & 
\end{array}$$

of  $(\Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta)[[G_{\mathbb{Q},S}]]$ -modules with exact rows and columns, where the maps  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta^l(1) \rightarrow \Lambda^l \hat{\otimes}_{\mathbb{Z}_p} Q$ . In it, we view  $f \in \Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta$  as acting on the leftmost two columns as  $w(f)$  and on the rightmost column by  $f$  (and in that sense  $1 \otimes \xi$  should be understood as the endomorphism induced by  $\tilde{\xi}$ ). In particular, this provides a canonical isomorphism

$$Q^\dagger \cong (\Lambda \hat{\otimes}_{\mathbb{Z}_p} Q) / \xi(\Lambda \hat{\otimes}_{\mathbb{Z}_p} Q)$$

of  $(\Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta)[[G_{\mathbb{Q},S}]]$ -modules, again understanding that  $f \in \Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta$  acts on the right by  $w(f)$ . As  $\Lambda$  is  $\mathbb{Z}_p$ -free with trivial  $G_{\mathbb{Q},S}$ -action, we have

$$H^i(\mathcal{O}, \Lambda \hat{\otimes}_{\mathbb{Z}_p} Q(1)) \cong \Lambda \hat{\otimes}_{\mathbb{Z}_p} H^i(\mathcal{O}, Q(1))$$

for all  $i$ . So, we have exact sequences

$$\begin{aligned} \cdots \rightarrow \Lambda \hat{\otimes}_{\mathbb{Z}_p} H^i(\mathcal{O}, Q(1)) \xrightarrow{\xi} \Lambda \hat{\otimes}_{\mathbb{Z}_p} H^i(\mathcal{O}, Q(1)) \rightarrow H^i(\mathcal{O}, Q^\dagger(1)) \\ \rightarrow \Lambda \hat{\otimes}_{\mathbb{Z}_p} H^{i+1}(\mathcal{O}, Q(1)) \xrightarrow{\xi} \Lambda \hat{\otimes}_{\mathbb{Z}_p} H^{i+1}(\mathcal{O}, Q(1)) \rightarrow \cdots \end{aligned}$$

of  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta$ -modules for all  $i$ . As  $\xi$  is a unit times a distinguished polynomial in  $\Lambda_\theta[[X]]$  and the leading coefficient of a distinguished polynomial is 1, the endomorphism  $\xi$  has no kernel on the modules  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} H^i(\mathcal{O}, Q(1))$ . Hence, the exact sequence provides the result.  $\square$

Consider the exact sequences

$$0 \rightarrow Q^\dagger \xrightarrow{X} (\Lambda^t \hat{\otimes}_{\mathbb{Z}_p} Q)/X\xi \rightarrow Q \rightarrow 0 \quad (3.4)$$

$$0 \rightarrow Q \xrightarrow{\xi} (\Lambda^t \hat{\otimes}_{\mathbb{Z}_p} Q)/X\xi \rightarrow Q^\dagger \rightarrow 0 \quad (3.5)$$

of  $\Lambda[[G_{Q,S}]]$ -modules, with the action of  $\Lambda$  induced from the action on  $Q$ . For the latter sequence, note that multiplication by  $\xi$  induces an isomorphism

$$Q \xrightarrow{\sim} \xi(\Lambda^t \hat{\otimes}_{\mathbb{Z}_p} Q)/X\xi(\Lambda^t \hat{\otimes}_{\mathbb{Z}_p} Q).$$

We similarly have exact sequences

$$0 \rightarrow (\Lambda^t \hat{\otimes}_{\mathbb{Z}_p} Q)/\xi_1 \xrightarrow{X} Q^\dagger \rightarrow Q \rightarrow 0 \quad (3.6)$$

$$0 \rightarrow Q \xrightarrow{\xi_1} Q^\dagger \rightarrow (\Lambda^t \hat{\otimes}_{\mathbb{Z}_p} Q)/\xi_1 \rightarrow 0, \quad (3.7)$$

where  $X\xi_1 = \xi - 1 \otimes \xi$ .

The following refines Proposition 9.3.3 of [FK] in our case of interest.

**Proposition 3.3.4.** *Let  $\partial_Q^\dagger$  denote the composition*

$$H^1(\mathcal{O}, Q(1)) \rightarrow H^2(\mathcal{O}, \Lambda_\theta^t(2)) \rightarrow H^2(\mathcal{O}, Q(1)),$$

*of the isomorphisms of Lemma 3.1.4. Then  $\partial_Q^\dagger$  fits in a commutative diagram*

$$\begin{array}{ccc} H^1(\mathcal{O}, Q^\dagger(1)) & \longrightarrow & H^2(\mathcal{O}, Q(1)) \\ \downarrow & \nearrow \partial_Q^\dagger & \downarrow \\ H^1(\mathcal{O}, Q(1)) & \longrightarrow & H^2(\mathcal{O}, Q^\dagger(1)) \end{array}$$

*in which the horizontal maps are connecting homomorphisms of (3.5) and (3.4) and the vertical maps are induced from maps in (3.6) and (3.7) and are surjective and injective, respectively.*

*Proof.* The commutativity of the outside square follows from the morphism of exact sequences

$$\begin{array}{ccccccc}
0 & \longrightarrow & Q & \xrightarrow{\xi} & (\Lambda^1 \hat{\otimes}_{\mathbb{Z}_p} Q)/X\xi & \longrightarrow & Q^\dagger \longrightarrow 0 \\
& & \downarrow \xi_1 & & \parallel & & \downarrow \\
0 & \longrightarrow & Q^\dagger & \xrightarrow{X} & (\Lambda^1 \hat{\otimes}_{\mathbb{Z}_p} Q)/X\xi & \longrightarrow & Q \longrightarrow 0
\end{array}$$

By Proposition 3.3.3, the left-hand vertical map is identified with the reduction modulo  $X$  map

$$(\Lambda \hat{\otimes}_{\mathbb{Z}_p} H^1(\mathcal{O}, Q(1)))/\xi \twoheadrightarrow H^1(\mathcal{O}, Q(1)), \quad (3.8)$$

and the right-hand vertical map is identified with the canonical injection

$$H^2(\mathcal{O}, Q(1)) \hookrightarrow (\Lambda \hat{\otimes}_{\mathbb{Z}_p} H^2(\mathcal{O}, Q(1)))/\xi,$$

given by  $w(\xi_1)$ .

It is sufficient to verify the commutativity of the upper triangle in the diagram of the proposition. The commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Lambda_\theta^1(1) & \longrightarrow & \Lambda_\theta^1(1) & \longrightarrow & Q \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \Lambda_\theta^1(1) & \xrightarrow{1 \otimes \xi} & (\Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta^1(1))/(X \otimes \xi) & \longrightarrow & \Lambda \hat{\otimes}_{\mathbb{Z}_p} Q \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Q & \longrightarrow & (\Lambda^1 \hat{\otimes}_{\mathbb{Z}_p} Q)/X\xi & \longrightarrow & (\Lambda^1 \hat{\otimes}_{\mathbb{Z}_p} Q)/\xi \longrightarrow 0
\end{array}$$

with exact rows gives rise to a commutative diagram

$$\begin{array}{ccc}
H^1(\mathcal{O}, Q(1)) & \xrightarrow{\sim} & H^2(\mathcal{O}, \Lambda_\theta^1(2)) \\
\uparrow \text{mod } X & & \parallel \\
\Lambda \hat{\otimes}_{\mathbb{Z}_p} H^1(\mathcal{O}, Q(1)) & \twoheadrightarrow & H^2(\mathcal{O}, \Lambda_\theta^1(2)) \\
\downarrow & & \downarrow \wr \\
H^1(\mathcal{O}, Q^\dagger(1)) & \longrightarrow & H^2(\mathcal{O}, Q(1)),
\end{array}$$

where the composition  $H^1(\mathcal{O}, Q(1)) \rightarrow H^2(\mathcal{O}, Q(1))$  is  $\partial_Q^\dagger$ . Noting (3.8), the upper triangle commutes.  $\square$

**Proposition 3.3.5.** *We have a commutative square of isomorphisms*

$$\begin{array}{ccc} H_c^2(\mathcal{O}, P(1)) & \xrightarrow{\sim} & H_c^3(\mathcal{O}, P^\dagger(1)) \\ \downarrow \wr & \searrow \partial_P^\dagger & \downarrow \wr \\ H_c^2(\mathcal{O}, P^\dagger(1)) & \xrightarrow{\sim} & H_c^3(\mathcal{O}, P(1)), \end{array}$$

between  $\Lambda$ -modules canonically isomorphic to  $P$ , in which every vertical and horizontal map is identified with the identity map on  $P$ . The same holds with  $\mathcal{O}$  replaced with  $\mathbb{Z}[\frac{1}{p}]$ .

*Proof.* First, note that the diagram commutes as in the proof of Proposition 3.3.4. We have

$$H_c^3(\mathcal{O}, P^\dagger(1)) \cong H_c^3(\mathcal{O}, \Lambda^1 \hat{\otimes}_{\mathbb{Z}_p} P(1)) / \tilde{\xi} \cong P / \tilde{\xi} P \cong P,$$

as compactly supported cohomology has  $p$ -cohomological dimension 3, and an exact sequence

$$\begin{aligned} \cdots \rightarrow H_c^2(\mathcal{O}, \Lambda^1 \hat{\otimes}_{\mathbb{Z}_p} P(1)) \xrightarrow{\tilde{\xi}} H_c^2(\mathcal{O}, \Lambda^1 \hat{\otimes}_{\mathbb{Z}_p} P(1)) \rightarrow H_c^2(\mathcal{O}, P^\dagger(1)) \\ \rightarrow H_c^3(\mathcal{O}, \Lambda^1 \hat{\otimes}_{\mathbb{Z}_p} P(1)) \xrightarrow{\tilde{\xi}} H_c^3(\mathcal{O}, \Lambda^1 \hat{\otimes}_{\mathbb{Z}_p} P(1)) \rightarrow \cdots \end{aligned}$$

Note that  $H_c^2(\mathcal{O}, \Lambda^1 \hat{\otimes}_{\mathbb{Z}_p} P(1))$  is isomorphic to the tensor product with  $P$  of the Galois group of the maximal abelian pro- $p$ ,  $S$ -ramified extension of  $\mathbb{Q}_\infty$ , which is trivial (since no prime dividing  $N$  is 1 modulo  $p$ ), so the first two terms are zero. The last map is also zero since multiplication by  $\tilde{\xi}$  is trivial on  $P$ . Thus, we have

$$H_c^2(\mathcal{O}, P^\dagger(1)) \xrightarrow{\sim} H_c^3(\mathcal{O}, \Lambda^1 \hat{\otimes}_{\mathbb{Z}_p} P(1)) \xrightarrow{\sim} H_c^3(\mathcal{O}, P(1)).$$

We choose the identification of  $H_c^2(\mathcal{O}, P^\dagger(1))$  with  $P$  which makes this the identity map, and the right-hand vertical map is identified with the identity map on  $P$  via invariant maps. As for the upper map, note that it factors as

$$H_c^2(\mathcal{O}, P(1)) \rightarrow H_c^3(\mathcal{O}, \Lambda^1 \hat{\otimes}_{\mathbb{Z}_p} P(1)) \rightarrow H_c^3(\mathcal{O}, P^\dagger(1)),$$

where the first map is the connecting homomorphism, which is seen to be the identity map by using Poitou-Tate duality as in Lemma 3.1.16, and the second map is again clearly identified with the identity map on  $P$ . The same argument works with  $\mathcal{O}$  replaced by  $\mathbb{Z}[\frac{1}{p}]$ .  $\square$

The following result is a special case of the anticommutativity of connecting homomorphisms for a commutative square of short exact sequences of complexes.



**Lemma 3.3.6.** *The diagram of connecting homomorphisms*

$$\begin{array}{ccc} H^1(\mathcal{O}, Q^\dagger(1)) & \xrightarrow{\partial_Q^\dagger} & H^2(\mathcal{O}, Q(1)) \\ \downarrow \Theta^\dagger & & \downarrow \Upsilon \\ H_c^2(\mathcal{O}, P^\dagger(1)) & \xrightarrow{\partial_P^\dagger} & H_c^3(\mathcal{O}, P(1)), \end{array}$$

is anticommutative.

By Proposition 3.3.4, the upper horizontal map in the diagram of Lemma 3.3.6 factors as

$$H^1(\mathcal{O}, Q^\dagger(1)) \rightarrow H^1(\mathcal{O}, Q(1)) \xrightarrow{\partial_Q^\dagger} H^2(\mathcal{O}, Q(1)),$$

and these maps are identified with

$$(\Lambda \hat{\otimes}_{\mathbb{Z}_p} Y) / \xi (\Lambda \hat{\otimes}_{\mathbb{Z}_p} Y) \rightarrow Y \xrightarrow{\text{id}} Y,$$

where the first map is the quotient map. Together with Proposition 3.3.5, it then follows that  $\Theta^\dagger$  factors as

$$H^1(\mathcal{O}, Q^\dagger(1)) \rightarrow H^1(\mathcal{O}, Q(1)) \xrightarrow{\Phi} H_c^2(\mathcal{O}, P(1)) \xrightarrow{\sim} H_c^2(\mathcal{O}, P^\dagger(1))$$

for some map  $\Phi$  satisfying  $-\partial_P^\dagger \circ \Phi = \Upsilon \circ \partial_Q^\dagger$ , which we may also view as a map  $\Phi: Y \rightarrow P$ .

**Lemma 3.3.7.** *The connecting homomorphism  $H^1(\mathcal{O}, Q^\dagger(1)) \rightarrow H^2(\mathcal{O}, P^\dagger(1))$  is zero.*

*Proof.* Consider the commutative diagram

$$\begin{array}{ccc} H^1(\mathcal{O}, Q^\dagger(1)) & \longrightarrow & H^2(\mathcal{O}, P^\dagger(1)) \\ \downarrow & & \downarrow \\ \bigoplus_{\ell|Np} H^1(\mathbb{Q}_\ell, Q^\dagger(1)) & \longrightarrow & \bigoplus_{\ell|Np} H^2(\mathbb{Q}_\ell, P^\dagger(1)) \\ \downarrow & & \downarrow \\ H_c^2(\mathcal{O}, Q^\dagger(1)) & \xrightarrow{\sim} & H_c^3(\mathcal{O}, P^\dagger(1)) \end{array} \quad (3.9)$$

with exact columns. The right-hand column is isomorphic to the quotient by  $\xi$  of the middle terms of the short exact sequence

$$0 \rightarrow H_{\text{Iw}}^2(\mathcal{O}_\infty, P(1)) \rightarrow \bigoplus_{\ell|Np} H_{\text{Iw}}^2(\mathbb{Q}_{\ell, \infty}, P(1)) \rightarrow H_{c, \text{Iw}}^3(\mathcal{O}_\infty, P(1)) \rightarrow 0,$$

where the first map is injective since  $H_{c, \text{Iw}}^2(\mathcal{O}_\infty, P(1)) = 0$ . In the second sum, since any prime  $\ell$  dividing  $N$  is inert in  $\mathbb{Q}_\infty$  and  $p$  is totally ramified, we have  $H_{\text{Iw}}^2(\mathbb{Q}_{\ell, \infty}, P(1)) \cong P$  for every  $\ell \mid Np$ . The third term is also isomorphic to  $P$  via the invariant map. As these groups are killed by  $\tilde{\xi}$ , the sequence remains exact upon taking the quotient by the action of  $\tilde{\xi}$ , and the map

$$H^2(\mathcal{O}, P^\dagger(1)) \rightarrow \bigoplus_{\ell \mid N} H^2(\mathbb{Q}_\ell, P^\dagger(1))$$

is an isomorphism. By the diagram (3.9), it therefore suffices to show that  $H^1(\mathbb{Q}_\ell, Q^\dagger(1)) = 0$  for all primes  $\ell \mid N$ . We verify this claim.

Let  $K_\ell = \mathbb{Q}_\ell(\mu_{Np^\infty})$ , and set  $\tilde{\Gamma}_\ell = \text{Gal}(K_\ell/\mathbb{Q}_\ell)$ ,

$$\Delta_\ell = \text{Gal}(K_\ell/\mathbb{Q}_{\ell, \infty}), \quad \text{and} \quad \Gamma_\ell = \text{Gal}(K_\ell/\mathbb{Q}_\ell(\mu_{Np})).$$

Inflation-restriction provides an exact sequence

$$0 \rightarrow H^1(\tilde{\Gamma}_\ell, Q^\dagger(1)) \rightarrow H^1(\mathbb{Q}_\ell, Q^\dagger(1)) \rightarrow H^1(K_\ell, Q^\dagger(1))^{\tilde{\Gamma}_\ell}.$$

We have  $H^1(K_\ell, Q^\dagger(1)) \cong Q^\dagger$  by Kummer theory and the valuation map (since all roots of unity are infinitely divisible by  $p$  in  $K_\ell^\times$ ). As  $\Delta_\ell$  acts on  $Q^\dagger$  through the restriction of  $\theta^{-1}$ , the  $\Delta_\ell$ -invariants of  $Q^\dagger$  are trivial by Hypothesis 2.4.3b. So, we have  $H^1(K_\ell, Q^\dagger(1))^{\tilde{\Gamma}_\ell} = 0$ . Moreover, since  $\Delta_\ell$  has prime-to- $p$  order, inflation provides an isomorphism

$$H^1(\Gamma_\ell, (Q^\dagger(1))^{\Delta_\ell}) \xrightarrow{\sim} H^1(\tilde{\Gamma}_\ell, Q^\dagger(1)),$$

and again the inertia subgroup of  $\Delta_\ell$  acts nontrivially on  $Q^\dagger(1)$  by assumption.  $\square$

**Lemma 3.3.8.** *The connecting homomorphisms  $H^1(\mathbb{Q}_\ell, P^\dagger(1)) \rightarrow H_{\text{Iw}}^2(\mathbb{Q}_{\ell, \infty}, P(1))$  for  $\ell \mid N$  are all isomorphisms.*

*Proof.* We have an exact sequence

$$H_{\text{Iw}}^1(\mathbb{Q}_{\ell, \infty}, P(1)) \rightarrow H^1(\mathbb{Q}_\ell, P^\dagger(1)) \rightarrow H_{\text{Iw}}^2(\mathbb{Q}_{\ell, \infty}, P(1)) \xrightarrow{\tilde{\xi}} H_{\text{Iw}}^2(\mathbb{Q}_{\ell, \infty}, P(1)),$$

and  $H_{\text{Iw}}^1(\mathbb{Q}_{\ell, \infty}, P(1)) = 0$  since  $\ell$  is unramified in  $\mathbb{Q}_{\ell, \infty}$ , while  $H_{\text{Iw}}^2(\mathbb{Q}_{\ell, \infty}, P(1)) \cong P$  via the invariant map. Since  $\tilde{\xi}$  acts as  $\xi$  on  $P$ , and  $\xi$  kills  $P$ , we have the result.  $\square$

**Proposition 3.3.9.** *The square*

$$\begin{array}{ccc} H^1(\mathcal{O}, T^\dagger(1)) & \longrightarrow & H^1(\mathcal{O}, Q^\dagger(1)) \\ \downarrow & & \downarrow -\Theta^\dagger \\ \bigoplus_{\ell \mid Np} H^1(\mathbb{Q}_\ell, P^\dagger(1)) & \longrightarrow & H_c^2(\mathcal{O}, P^\dagger(1)) \end{array}$$

*is commutative.*

*Proof.* Applying Lemma 3.3.7, we have a diagram

$$\begin{array}{ccccccc}
& & & & & & H^1(\mathcal{O}, Q^\dagger(1)) \\
& & & & & & \parallel \\
0 & \longrightarrow & H^1(\mathcal{O}, P^\dagger(1)) & \longrightarrow & H^1(\mathcal{O}, T^\dagger(1)) & \longrightarrow & H^1(\mathcal{O}, Q^\dagger(1)) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & \bigoplus_{\ell|Np} H^1(\mathbb{Q}_\ell, P^\dagger(1)) & = & \bigoplus_{\ell|Np} H^1(\mathbb{Q}_\ell, P^\dagger(1)) & & \\
& & \downarrow & & & & \\
& & H_c^2(\mathcal{O}, P^\dagger(1)) & & & & 
\end{array}$$

with exact rows and columns. The snake lemma map from the diagram is then the negative of the connecting homomorphism  $\Theta^\dagger$  by a standard lemma.  $\square$

We now have that all squares in the diagram

$$\begin{array}{ccccccc}
H^1(\mathcal{O}, T^\dagger(1)) & \longrightarrow & H^1(\mathcal{O}, Q^\dagger(1)) & \longrightarrow & H^1(\mathcal{O}, Q(1)) & \xrightarrow{\partial_Q^\dagger} & H^2(\mathcal{O}, Q(1)) \\
\downarrow & & \downarrow -\Theta^\dagger & & \downarrow -\Phi & & \downarrow \gamma \\
\bigoplus_{\ell|Np} H^1(\mathbb{Q}_\ell, P^\dagger(1)) & \longrightarrow & H_c^2(\mathcal{O}, P^\dagger(1)) & \xleftarrow{\sim} & H_c^2(\mathcal{O}, P(1)) & \xrightarrow{\partial_P^\dagger} & H_c^3(\mathcal{O}, P(1))
\end{array}$$

are commutative.

## 4 Local study

In this section, we let  $\mathfrak{A}$  denote a complete Noetherian semi-local  $\mathbb{Z}_p$ -algebra. We let  $A$  denote an unramified  $\mathfrak{A}[[G_{\mathbb{Q}_p}]]$ -module. Exactly when discussing this general setting, we shall allow  $p$  to be any prime.

### 4.1 Coleman maps

Let  $\mathcal{U}_\infty^{\text{ur}}$  (resp.,  $\mathcal{K}_\infty^{\text{ur}}$ ) denote the  $p$ -completion of the group of norm compatible sequences of units (resp., of nonzero elements) in the tower given by the cyclotomic  $\mathbb{Z}_p$ -extension  $\mathbb{Q}_{p,\infty}^{\text{ur}}$  of  $\mathbb{Q}_p^{\text{ur}}$ .

**Definition 4.1.1.** The *Coleman map*  $\text{Col}: \mathcal{K}_\infty^{\text{ur}} \rightarrow X^{-1}W[[X]]$  is the unique map of  $\Lambda$ -modules restricting to a map  $\mathcal{U}_\infty^{\text{ur}} \rightarrow W[[X]] = W[[1 + p\mathbb{Z}_p]]$  defined on  $(u_r)_{r \geq 1} \in \mathcal{U}_\infty^{\text{ur}}$  with  $u_r \in \mathbb{Q}_{p,r}^{\text{ur}}$  by

$$[\text{Col}((u_r)_{r \geq 1})](x) = \left(1 - \frac{\Psi}{p}\right) \log(f(x)).$$

Here,  $W[[X]]$  acts continuously and  $W$ -linearly on  $W[[x]]$  with the result of  $h \in W[[X]]$  acting on  $x$  denoted by  $[h](x)$ , via the action determined by  $[a](x) = x^a \in W[[x]]$  for  $a \in 1 + p\mathbb{Z}_p$ . Also,  $f(x) \in W[[x-1]]$  is the Coleman power series with  $f(\zeta_{p^r}) = \text{Fr}_p^r(u_r)$  for all  $r$ , and  $\psi$  is defined on  $g(x) \in W[[x]]$  by  $\psi(g)(x) = \text{Fr}_p(g)(x^p)$ .

We can extend this definition as follows.

**Definition 4.1.2.** The Coleman map for  $A$  is the map

$$\text{Col}_A : H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, A(1)) \rightarrow X^{-1}D(A)[[X]]$$

defined as the composition

$$H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, A(1)) \xrightarrow{\text{Inf}} (A \hat{\otimes}_{\mathbb{Z}_p} \mathcal{K}_\infty^{\text{ur}})^{\text{Fr}_p=1} \xrightarrow{1 \otimes \text{Col}} (A \hat{\otimes}_{\mathbb{Z}_p} X^{-1}W[[X]])^{\text{Fr}_p=1} \xrightarrow{\sim} X^{-1}D(A)[[X]],$$

where  $\text{Fr}_p$  acts diagonally on the tensor products.

The following is a slight extension, allowing  $A^{\text{Fr}_p=1}$  to be nonzero, of the restriction of [FK, 4.2.7] to invariants for  $\Delta \cong \text{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_{p,\infty})$ . Note that  $\text{Col}_A$  agrees with the map denoted  $\text{Col}$  in [FK] on the fixed part under  $\text{Gal}(\mathbb{Q}_p(\mu_p)/\mathbb{Q}_p)$ .

**Lemma 4.1.3.** *The map  $\text{Col}_A$  is injective with image in  $X^{-1}D(A)[[X]]$  equal to*

$$\mathfrak{C}(A) = X^{-1}A^{\text{Fr}_p=1} + D(A)[[X]].$$

*Proof.* Since  $A(1)$  has no  $G_{\mathbb{Q}_p^{\text{ur}}}$ -fixed part and  $\text{Gal}(\mathbb{Q}_p^{\text{ur}}/\mathbb{Q}_{p,\infty}) \cong \hat{\mathbb{Z}}$  has cohomological dimension 1, the inflation map  $\text{Inf}$  in the definition of  $\text{Col}_A$  is an isomorphism. It is well-known that the Coleman map  $\text{Col}$  is injective and, as follows for instance from the proof of [FK, 4.2.7], it restricts to an isomorphism  $\mathcal{K}_\infty^{\text{ur}} \xrightarrow{\sim} W[[X]]$ . In particular,  $\text{Col}_A$  is injective.

It follows that we have an exact sequence

$$0 \rightarrow A \hat{\otimes}_{\mathbb{Z}_p} W[[X]] \rightarrow A \hat{\otimes}_{\mathbb{Z}_p} \mathcal{K}_\infty^{\text{ur}} \rightarrow A \rightarrow 0$$

with the first map the inverse of  $1 \otimes \text{Col}$  and the second determined by the valuation map on the norm to  $\mathbb{Q}_p^{\text{ur}}$  of an element of  $\mathcal{K}_\infty^{\text{ur}}$ . The kernel of  $1 - \text{Fr}_p$  applied to this sequence gives

$$0 \rightarrow D(A)[[X]] \rightarrow H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, A(1)) \rightarrow A^{\text{Fr}_p=1} \rightarrow 0,$$

the surjectivity since  $\mathcal{K}_\infty^{\text{ur}}$  contains the Frobenius fixed sequence that is the projection of  $(1 - \zeta_{p^n})_n$  to the  $\Delta$ -invariant group. By the injectivity of  $\text{Col}_A$ , this forces the induced map  $A^{\text{Fr}_p=1} \rightarrow X^{-1}D(A)[[X]]/D(A)[[X]]$  to have image  $X^{-1}A^{\text{Fr}_p=1}$ . Since the image of  $\text{Col}_A$  contains  $D(A)[[X]]$ , it must then equal  $\mathfrak{C}(A)$ .  $\square$

*Remark 4.1.4.* If  $A^{\text{Fr}_p=1} = 0$ , then  $\text{Col}_A$  is an isomorphism  $H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, A(1)) \rightarrow D(A)[[X]]$ . This occurs, for instance, for  $A = \mathcal{T}_{\text{quo}}$  by [FK, 3.3.3].

In addition to  $\text{Col}_A$ , we also have a homomorphism at the level of  $\mathbb{Q}_p$  that can be defined as follows, following [FK, 4.2.2].

**Definition 4.1.5.** We let

$$\text{Col}_A^b: H^1(\mathbb{Q}_p, A(1)) \rightarrow D(A)$$

denote the composition

$$H^1(\mathbb{Q}_p, A(1)) \rightarrow (A \hat{\otimes}_{\mathbb{Z}_p} (1 + pW))^{\text{Fr}_p=1} \xrightarrow{\left(1 - \frac{\varphi}{p}\right) \log} (A \hat{\otimes}_{\mathbb{Z}_p} W)^{\text{Fr}_p=1} \xrightarrow{\sim} D(A),$$

where the first map is induced by restriction and the map

$$H^1(\mathbb{Q}_p^{\text{ur}}, \mathbb{Z}_p(1)) \xrightarrow{\sim} p^{\mathbb{Z}_p} \times (1 + pW) \rightarrow 1 + pW$$

is given by projection to the second coordinate.

*Remark 4.1.6.* The map  $\text{Col}_A^b$  is in general only split surjective, with a canonical splitting given by the valuation map

$$H^1(\mathbb{Q}_p, A(1)) \xrightarrow{\text{Inf}} H^1(\mathbb{Q}_p^{\text{ur}}, A(1))^{\text{Fr}_p=1} \xrightarrow{v_p} A^{\text{Fr}_p=1}.$$

Recall that

$$H^2(\mathbb{Q}_p, A(1)) \cong A/(\text{Fr}_p - 1)A \cong D(A)/(\varphi - 1)D(A) \quad (4.1)$$

via Tate duality (i.e., the invariant map of local class field theory). The following is [FK, 4.2.4].

**Lemma 4.1.7.** *The composition of  $\text{Col}_A^b$  with the quotient map  $D(A) \rightarrow D(A)/(\varphi - 1)D(A)$  is identified through (4.1) with the connecting homomorphism*

$$\partial_A: H^1(\mathbb{Q}_p, A(1)) \rightarrow H^2(\mathbb{Q}_p, A(1)).$$

*Proof.* By replacing  $A$  by  $A/(\text{Fr}_p - 1)A$ , we may suppose that  $A$  has trivial Galois action, and it then suffices to consider  $A = \mathbb{Z}_p$ . The connecting homomorphism  $\partial_A$  is given by left cup product with  $-\chi$  by Remark 3.1.14. Note that for  $a \in \mathbb{Q}_p^\times$ , we have  $\chi \cup a = \chi(\rho(a))$ , where  $\rho: \mathbb{Q}_p^\times \rightarrow G_{\mathbb{Q}_p}^{\text{ab}}$  is the local reciprocity map (cf. [Se, Chapter XIV, Propositions 1.3 and 2.5]). But

$$\rho(u)(\zeta_{p^n}) = \zeta_{p^n}^{u^{-1}}$$

for  $u \in 1 + p\mathbb{Z}_p$  and  $\rho(p)(\zeta_{p^n}) = \zeta_{p^n}$ . Then  $-\chi(\rho(p)) = 0$  and  $-\chi(\rho(u)) = (1 - p^{-1})\log(u)$  for  $u \in 1 + p\mathbb{Z}_p$ . Thus,  $\partial_{\mathbb{Z}_p} = \text{Col}_{\mathbb{Z}_p}^b$ .  $\square$

The relationship between  $\text{Col}$  and  $\text{Col}^b$  is given by the following [FK, 4.2.9].

**Proposition 4.1.8.** *Let  $\text{ev}_0: D(A)[[X]] \rightarrow D(A)$  denote evaluation at 0, and let  $\text{cor}$  be the corestriction map  $H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, A(1)) \rightarrow H^1(\mathbb{Q}_p, A(1))$ . Then we have*

$$\text{ev}_0 \circ (1 - \varphi^{-1}) \text{Col}_A = \text{Col}_A^b \circ \text{cor}$$

as maps  $H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, A(1)) \rightarrow D(A)$ .

## 4.2 Intermediate Coleman maps

In this subsection, we aim to construct a map  $\text{Col}_A^\dagger$  that plays an analogous role to  $\text{Col}_A^b$  for a certain quotient of  $\Lambda^l \hat{\otimes}_{\mathbb{Z}_p} A$ .

We suppose that  $\mathfrak{R}$  is local to simplify the discussion and fix an element

$$\alpha \in \Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{R} = \mathfrak{R}[[X]]$$

with nonzero image in  $\mathfrak{k}[[X]]$  for  $\mathfrak{k}$  the residue field of  $\mathfrak{R}$ . The multiplication-by- $\alpha$  map is then injective on  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} A$ .

**Definition 4.2.1.** The *intermediate quotient*  $A^\dagger$  of  $\Lambda^l \hat{\otimes}_{\mathbb{Z}_p} A$  is the  $(\Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{R})[[G_{\mathbb{Q}_p}]]$ -module defined as

$$A^\dagger = (\Lambda^l \hat{\otimes}_{\mathbb{Z}_p} A) / X\alpha(\Lambda^l \hat{\otimes}_{\mathbb{Z}_p} A).$$

By Weierstrass preparation,  $A^\dagger$  is a finite direct sum of copies of  $A$  as an  $\mathfrak{R}$ -module.

**Definition 4.2.2.**

- a. Set  $\mathfrak{C}^\dagger(A) = \mathfrak{C}(A) / X\alpha\mathfrak{C}(A)$ , where  $\mathfrak{C}(A)$  is the image of  $\text{Col}_A$ , as in Lemma 4.1.3.
- b. Let  $\mathfrak{C}^\star(A)$  denote the pushout of the diagram

$$\mathfrak{C}^\dagger(A) \xleftarrow{\alpha} D(A) \xrightarrow{1-\varphi^{-1}} D(A),$$

where the first map sends  $a \in D(A)$  to  $\alpha(1 \otimes a) \in \mathfrak{C}^\dagger(A)$ .

- c. Let

$$\bar{A} = A / (\text{Fr}_p - 1)A \cong D(A) / (\varphi - 1)D(A).$$

The pushout  $\mathfrak{C}^\star(A)$  has a relatively simple explicit description in the case  $A^{\text{Fr}_p=1} = 0$ , noting that  $1 - \varphi^{-1}$  is then injective on  $D(A)$ .

**Lemma 4.2.3.** *If  $A^{\text{Fr}_p=1} = 0$ , then*

$$\mathfrak{C}^*(A) \cong \frac{(1 - \varphi, \alpha) \Lambda \hat{\otimes}_{\mathbb{Z}_p} D(A)}{X\alpha(\Lambda \hat{\otimes}_{\mathbb{Z}_p} D(A))}.$$

Moreover, the injective pushout map from

$$\mathfrak{C}^\dagger(A) = \frac{\Lambda \hat{\otimes}_{\mathbb{Z}_p} D(A)}{X\alpha(\Lambda \hat{\otimes}_{\mathbb{Z}_p} D(A))}$$

to  $\mathfrak{C}^*(A)$  is given by multiplication by  $1 - \varphi^{-1}$ .

The following defines an *intermediate Coleman map* from  $H^1(\mathbb{Q}_p, A^\dagger(1))$  to  $\mathfrak{C}^*(A)$ .

**Theorem 4.2.4.** *There is an isomorphism*

$$\text{Col}_A^\dagger : H^1(\mathbb{Q}_p, A^\dagger(1)) \xrightarrow{\sim} \mathfrak{C}^*(A)$$

fitting in an isomorphism of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, A(1))/X\alpha & \longrightarrow & H^1(\mathbb{Q}_p, A^\dagger(1)) & \longrightarrow & H_{\text{Iw}}^2(\mathbb{Q}_{p,\infty}, A(1)) \longrightarrow 0 \\ & & \downarrow \text{Col}_A & & \downarrow \text{Col}_A^\dagger & & \downarrow \text{inv} \\ 0 & \longrightarrow & \mathfrak{C}^\dagger(A) & \longrightarrow & \mathfrak{C}^*(A) & \xrightarrow{\psi} & \bar{A} \longrightarrow 0, \end{array}$$

where the left lower horizontal map is given by the pushout, and  $\psi$  is inverse to the isomorphism  $\bar{A} \rightarrow \mathfrak{C}^*(A)/\mathfrak{C}^\dagger(A)$  induced by the other pushout map.

*Proof.* Consider the composition  $D(A) \rightarrow H^1(\mathbb{Q}_p, A^\dagger(1))$  of the canonical splitting

$$(\text{Col}_A^b)^{-1} : D(A) \rightarrow H^1(\mathbb{Q}_p, A(1))$$

of  $\text{Col}_A^b$  arising from Remark 4.1.6 and the map  $H^1(\mathbb{Q}_p, A(1)) \rightarrow H^1(\mathbb{Q}_p, A^\dagger(1))$  induced by  $\alpha : A \rightarrow A^\dagger$ . We claim that the two compositions

$$D(A) \xrightarrow{1-\varphi^{-1}} D(A) \rightarrow H^1(\mathbb{Q}_p, A^\dagger(1)) \quad \text{and} \quad D(A) \xrightarrow{\alpha} \mathfrak{C}^\dagger(A) \xrightarrow{\text{Col}_A^{-1}} H^1(\mathbb{Q}_p, A^\dagger(1))$$

agree. Given the claim, we define  $\text{Col}_A^\dagger$  as the inverse of the map given by universal property of the pushout  $\mathfrak{C}^*(A)$ , and the left-hand square in the diagram of the proposition commutes.

To see the claim, consider the diagram

$$\begin{array}{ccccc} D(A) & \longrightarrow & \mathfrak{C}(A)/X\mathfrak{C}(A) & \xrightarrow{\alpha} & \mathfrak{C}^\dagger(A) \\ \downarrow 1-\varphi^{-1} & \swarrow 1-\varphi^{-1} & \downarrow \text{Col}_A^{-1} & & \downarrow \text{Col}_A^{-1} \\ D(A) & \xrightarrow{\quad} & H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, A(1))/X & \xrightarrow{\alpha} & H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, A(1))/X\alpha \\ & \searrow & \downarrow \text{cor} & & \downarrow \\ & & H^1(\mathbb{Q}_p, A(1)) & \xrightarrow{\alpha} & H^1(\mathbb{Q}_p, A^\dagger(1)) \end{array}$$

in which the two compositions are found by tracing its perimeter. The two right-hand squares clearly commute. Since the multiplication-by- $\alpha$  maps in this diagram are all injective, we are reduced to the commutativity of the left part of the diagram (aside from the dashed arrow). This commutativity follows from Proposition 4.1.8, which is equivalent to the statement that the two compositions  $\mathfrak{C}(A)/X\mathfrak{C}(A) \rightarrow H^1(\mathbb{Q}_p, A(1))$  agree on the image of  $D(A)$  in

$$\mathfrak{C}(A)/X\mathfrak{C}(A) \cong X^{-1}A^{\text{Fr}_p=1} \oplus D(A)/A^{\text{Fr}_p=1}.$$

The commutativity of the right-hand square in the map of exact sequences is seen as follows: we have the diagram

$$(4.2) \quad \begin{array}{ccccccc} & & & \text{inv} & & & \\ & & & \curvearrowright & & & \\ H^2(\mathbb{Q}_p, A(1)) & \xleftarrow{\partial_A} & H^1(\mathbb{Q}_p, A(1)) & \xrightarrow{\text{Col}_A^b} & D(A) & \xrightarrow{\quad} & \bar{A} \\ & \uparrow \wr^{\text{cor}} & \downarrow \alpha & & \downarrow \alpha & & \parallel \\ H_{\text{Iw}}^2(\mathbb{Q}_{p,\infty}, A(1)) & \xleftarrow{\partial_A^\dagger} & H^1(\mathbb{Q}_p, A^\dagger(1)) & \xrightarrow{\text{Col}_A^\dagger} & \mathfrak{C}^*(A) & \xrightarrow{\psi} & \bar{A} \\ & & & \curvearrowleft & & & \\ & & & \text{inv} & & & \end{array}$$

in which the map of rows and the outer part of the diagram commute by construction, and the top row commutes with the invariant map by Lemma 4.1.7. Consequently, the lower row commutes with the invariant map as well.  $\square$

*Remark 4.2.5.* The middle square of the commutative diagram (4.2) gives a comparison between  $\text{Col}_A^b$  and  $\text{Col}_A^\dagger$ . Note that in the case  $\alpha = 1$ , the map  $\text{Col}_A^b$  is defined as a split surjection (as we have kept the conventions of [FK]), whereas  $\text{Col}_A^\dagger$  is an isomorphism to  $A^{\text{Fr}_p=1} \oplus D(A)$ .

*Remark 4.2.6.* In [FK, Section 4], Coleman maps  $\text{Col}$  are defined on the Iwasawa cohomology of  $A(1)$  for the extension  $\mathbb{Q}_p(\mu_{p^\infty})$  of  $\mathbb{Q}_p$ , as opposed to just  $\mathbb{Q}_{p,\infty}$ . The second Iwasawa cohomology groups of  $A(1)$  for each of these extensions are isomorphic via corestriction. Outside of the trivial eigenspace for  $\text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}_{p,\infty})$  that we consider here, analogously defined intermediate Coleman maps would simply amount to reductions of the original Coleman maps.

Recall that the action of  $\varphi^{-1}$  on  $D(\mathcal{T}_{\text{quo}})$  agrees with the action of  $U_p$  on  $\mathfrak{S}_\theta$ . Given the identifications of Lemma 4.2.3, Theorem 4.2.4 then has the following corollary.

**Corollary 4.2.7.** *Set*

$$\mathfrak{S}_\theta^* = (\alpha, 1 - U_p)\Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{S}_\theta / X\alpha(\Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{S}_\theta) \subset \mathfrak{S}_\theta^\dagger = \Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{S}_\theta / X\alpha(\Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{S}_\theta).$$



There is an isomorphism

$$\text{Col}^\dagger : H^1(\mathbb{Q}_p, \mathcal{T}_{\text{quo}}^\dagger(1)) \rightarrow \mathfrak{S}_\theta^*$$

fitting in an isomorphism of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, \mathcal{T}_{\text{quo}}(1))/X\alpha & \longrightarrow & H^1(\mathbb{Q}_p, \mathcal{T}_{\text{quo}}^\dagger(1)) & \longrightarrow & H_{\text{Iw}}^2(\mathbb{Q}_{p,\infty}, \mathcal{T}_{\text{quo}}(1)) \longrightarrow 0 \\ & & \downarrow \text{Col} & & \downarrow \text{Col}^\dagger & & \downarrow \text{inv} \\ 0 & \longrightarrow & \mathfrak{S}_\theta^\dagger & \xrightarrow{1-U_p} & \mathfrak{S}_\theta^* & \xrightarrow{\psi} & \mathcal{S}_\theta \longrightarrow 0, \end{array}$$

where  $\psi$  factors through the inverse to the map induced by multiplication by  $\alpha$  on the cokernel of multiplication by  $1 - U_p$  on  $\mathfrak{S}_\theta^*$ .

We make the following definition for later use.

**Definition 4.2.8.** We let

$$\overline{\text{Col}}^\dagger : H^1(\mathbb{Q}_p, \mathcal{T}_{\text{quo}}^\dagger(1)) \rightarrow \mathcal{S}_\theta.$$

denote the composition  $\psi \circ \text{Col}^\dagger$ .

### 4.3 Local zeta maps

In this subsection, we use an ad hoc local version of the global zeta map of Fukaya-Kato. We shall see how it ties in with global elements in Section 5.

Fix an isomorphism  $\mathcal{M}_\theta \xrightarrow{\sim} \mathfrak{M}_\theta$  of  $\mathfrak{H}_\theta$  that reduces to the canonical isomorphism

$$\mathcal{M}_\theta / (U_p - 1)\mathcal{M}_\theta \xrightarrow{\sim} \mathfrak{M}_\theta / (U_p - 1)\mathfrak{M}_\theta.$$

We use it, in particular, to identify  $\mathcal{S}_\theta$  with  $\mathfrak{S}_\theta$  in the remainder of the paper. We then have isomorphisms

$$\text{Hom}_{\Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{h}_\theta}(\Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathcal{S}_\theta, \Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{S}_\theta) \xrightarrow{\sim} \text{End}_{\Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{h}_\theta}(\Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{S}_\theta) \xrightarrow{\sim} \Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{h}_\theta,$$

the second being the inverse of the map that takes an element to the endomorphism it defines.

We will specify the following element  $\alpha_\theta$  precisely in Section 4.

**Definition 4.3.1.** Let  $\alpha = \alpha_\theta \in \Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{h}_\theta$  denote an element with image equal to the image of  $\tilde{\xi}_1 = X^{-1}(\tilde{\xi} - 1 \otimes \xi)$  in the quotient ring

$$(\Lambda \hat{\otimes}_{\mathbb{Z}_p} (\mathfrak{h}/I)_\theta) / \tilde{\xi} (\Lambda \hat{\otimes}_{\mathbb{Z}_p} (\mathfrak{h}/I)_\theta).$$

We may then define a local zeta map. Its significance lies in that is induced by the restriction of a zeta map of Fukaya and Kato for our later good choice of  $\alpha_\theta$ .

**Definition 4.3.2.** Let  $z_{\text{quo}}$  denote the unique map of  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{h}_\theta$ -modules

$$z_{\text{quo}} : \Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathcal{S}_\theta \rightarrow H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, \mathcal{T}_{\text{quo}}(1))$$

such that  $\text{Col} \circ z_{\text{quo}}$  is identified with multiplication by  $\alpha_\theta \in \Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{h}_\theta$ .

The following is due to Fukaya and Kato (see [FK, 3.3.9, 4.3.8, 4.4.3, 8.1.2]).

**Proposition 4.3.3** (Fukaya-Kato). *There exists a unique  $\mathfrak{h}_\theta$ -module homomorphism*

$$z_{\text{quo}}^\sharp : \mathcal{S}_\theta \rightarrow H^1(\mathbb{Q}_p, \mathcal{T}_{\text{quo}}(1))$$

for which

$$(1 - U_p) z_{\text{quo}}^\sharp \circ \text{ev}_0 = \text{cor} \circ z_{\text{quo}}$$

on  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathcal{S}_\theta$ , and such that  $\text{Col}^\flat \circ z_{\text{quo}}^\sharp$  is multiplication by  $\xi'$  modulo  $I$ .

*Proof.* Since  $z_{\text{quo}}$  is defined so that  $\text{Col} \circ z_{\text{quo}}$  is multiplication by  $\alpha$  and  $(1 - U_p) \text{ev}_0 \circ \text{Col} = \text{Col}^\flat \circ \text{cor}$  by Proposition 4.1.8 (noting [FK, 1.8.1] to see that  $\varphi^{-1}$  acts as  $U_p$  on  $\mathcal{T}_{\text{quo}}$ ), we have that

$$\text{Col}^\flat \circ \text{cor} \circ z_{\text{quo}} = (1 - U_p) \text{ev}_0 \circ \text{Col} \circ z_{\text{quo}} = (1 - U_p) \alpha(0) \circ \text{ev}_0.$$

Since  $\text{Col}^\flat$  is an isomorphism for  $\mathcal{T}_{\text{quo}}$ , we can define  $z_{\text{quo}}^\sharp$  to be the unique map satisfying  $\text{Col}^\flat \circ z_{\text{quo}}^\sharp = \alpha(0)$ . As  $\alpha(0)$  modulo  $I$  is  $\tilde{\xi}_1(0) = \xi'$  by definition, we are done.  $\square$

We prove an analogue of Proposition 4.3.3 not involving the derivative  $\xi'$  for the intermediate quotient  $\mathcal{T}_{\text{quo}}^\dagger$ .

**Proposition 4.3.4.** *There exists a unique map*

$$z_{\text{quo}}^\dagger : \mathcal{S}_\theta \rightarrow H^1(\mathbb{Q}_p, \mathcal{T}_{\text{quo}}^\dagger(1))$$

of  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{h}_\theta$ -modules with the property that the square

$$\begin{array}{ccc} \Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathcal{S}_\theta & \xrightarrow{z_{\text{quo}}^\dagger \circ \text{ev}_0} & H^1(\mathbb{Q}_p, \mathcal{T}_{\text{quo}}^\dagger(1)) \\ \downarrow z_{\text{quo}} & & \downarrow 1 - U_p \\ H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, \mathcal{T}_{\text{quo}}(1)) & \longrightarrow & H^1(\mathbb{Q}_p, \mathcal{T}_{\text{quo}}^\dagger(1)) \end{array}$$

commutes, and the composition  $\overline{\text{Col}}^\dagger \circ z_{\text{quo}}^\dagger : \mathcal{S}_\theta \rightarrow S_\theta$  is reduction modulo  $(U_p - 1)$ .

*Proof.* Consider the composition

$$\Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathcal{S}_\theta \xrightarrow{z_{\text{quo}}} H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, \mathcal{T}_{\text{quo}}(1)) \rightarrow H^1(\mathbb{Q}_p, \mathcal{T}_{\text{quo}}^\dagger(1)) \xrightarrow{\text{Col}^\dagger} \mathfrak{S}_\theta^*.$$

By definition of  $\alpha$  and  $\text{Col}^\dagger$ , this map is induced by multiplication by  $(U_p - 1)\alpha$ . In particular, it factors through  $\mathcal{S}_\theta$  since it lands kernel of multiplication by  $X$  in  $\mathfrak{S}_\theta^*$ . So, we have the existence and uniqueness of  $z_{\text{quo}}^\dagger$  making the square commute and such that the composition

$$\mathcal{S}_\theta \xrightarrow{z_{\text{quo}}^\dagger} H^1(\mathbb{Q}_p, \mathcal{T}_{\text{quo}}^\dagger(1)) \xrightarrow{\text{Col}^\dagger} \mathfrak{S}_\theta^*$$

is induced by multiplication by  $\alpha$ . The composition of this map with  $\psi$  of Corollary 4.2.7 is reduction modulo  $(U_p - 1)$ , which gives the final statement.  $\square$

## 5 Global study

### 5.1 Global cohomology

We first consider torsion in global cohomology groups. As we are working only with the needed eigenspace of the Eisenstein part of cohomology, we can obtain finer results than [FK, Section 3] in our case of interest.

**Lemma 5.1.1.** *We have two exact sequences*

$$\begin{aligned} 0 \rightarrow H_{\text{Iw}}^1(\mathcal{O}_\infty, \mathcal{T}_\theta(1)) \rightarrow H_{\text{Iw}}^1(\mathcal{O}_\infty, \tilde{\mathcal{T}}_\theta(1)) \rightarrow H_{\text{Iw}}^1(\mathcal{O}_\infty, \tilde{\mathcal{T}}_\theta/\mathcal{T}_\theta(1)) \\ 0 \rightarrow H^1(\mathcal{O}, \mathcal{T}_\theta(1)) \rightarrow H^1(\mathcal{O}, \tilde{\mathcal{T}}_\theta(1)) \rightarrow H^1(\mathcal{O}, \tilde{\mathcal{T}}_\theta/\mathcal{T}_\theta(1)) \end{aligned}$$

of  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{S}_\theta$ -modules. In the first, the terms have no nonzero  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta$ -torsion, and in the second, they have no  $\Lambda_\theta$ -torsion.

*Proof.* The first sequence is automatically exact, as zeroth Iwasawa cohomology groups are trivial. Note that  $\tilde{\mathcal{T}}_\theta/\mathcal{T}_\theta$  has trivial  $G_{\mathbb{Q}(\mu_N)}$ -action by [FK, 3.2.4]. (Alternatively, one can see this by observing that the action factors through the Galois group of the totally ramified at  $p$  extension  $\mathbb{Q}(\mu_{Np^\infty})/\mathbb{Q}(\mu_N)$ , since all cusps of  $Y_1(Np^r)$  are defined over  $\mathbb{Q}(\mu_{Np^r})$ , and then that the  $G_{\mathbb{Q}_p}$ -action on  $\tilde{\mathcal{T}}_\theta/\mathcal{T}_\theta \cong \tilde{\mathcal{T}}_{\text{quo}}/\mathcal{T}_{\text{quo}}$  is unramified.) So, the second sequence is exact as  $H^0(\mathcal{O}, \tilde{\mathcal{T}}_\theta/\mathcal{T}_\theta(1)) = 0$ .

We can filter any  $\mathfrak{h}_\theta[G_{\mathbb{Q},S}]$ -subquotient  $M$  of  $\tilde{\mathcal{T}}_\theta$  by the powers of  $I$ , and we clearly have  $H^0(\mathcal{O}, M(1)) = 0$  if  $H^0(\mathcal{O}, I^k M/I^{k+1} M(1)) = 0$  for all  $k \geq 0$ . Let  $\mu \in \Lambda_\theta$  be nonzero, and set  $M = \tilde{\mathcal{T}}_\theta/\mu \tilde{\mathcal{T}}_\theta$ . As  $\tilde{\mathcal{T}}_\theta$  is  $\Lambda_\theta$ -free, we have an exact sequence

$$0 \rightarrow \tilde{\mathcal{T}}_\theta \xrightarrow{\mu} \tilde{\mathcal{T}}_\theta \rightarrow M \rightarrow 0,$$

so  $H^0(\mathcal{O}, M(1))$  surjects onto (in fact, is isomorphic to) the  $\mu$ -torsion in  $H^1(\mathcal{O}, \mathcal{T}_\theta(1))$ . Set  $T_k = I^k \mathcal{T}_\theta / I^{k+1} \mathcal{T}_\theta$ . Let  $P_k$  denote the  $\mathfrak{h}_\theta[G_{\mathbb{Q}, S}]$ -module that is the image of the multiplication map  $I^k \otimes_{\mathfrak{h}_\theta} P \rightarrow T_k$ . The  $G_{\mathbb{Q}}$ -action on  $P_k$  is then trivial, and on the quotient  $Q_k = T_k / P_k$ , the  $G_{\mathbb{Q}}$ -action factors through  $\mathbb{Z}_{p, N}^\times$  with  $\Delta$  acting as  $\omega\theta^{-1}$ . As a nonzero  $\mathfrak{h}_\theta[G_{\mathbb{Q}}]$ -subquotient of  $T_k(1)$ , it then follows (since  $\theta \neq \omega^2$  by Hypothesis 2.4.3c) that  $I^k M / I^{k+1} M(1)$  has no nonzero  $G_{\mathbb{Q}}$ -fixed elements. Thus,  $H^1(\mathcal{O}, \mathcal{T}_\theta(1))$  has no  $\mu$ -torsion. Replacing  $M$  with  $\Lambda^1 \hat{\otimes}_{\mathbb{Z}_p} M$  and  $\mu$  with a nonzero element  $\lambda \in \Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta$ , a similar argument applies to show that  $H^1(\mathcal{O}, \Lambda^1 \hat{\otimes}_{\mathbb{Z}_p} \mathcal{T}_\theta(1))$  has no nonzero  $\lambda$ -torsion (as the  $\Delta$ -action on  $\Lambda^1$  is trivial).

It remains to deal with the  $\tilde{\mathcal{T}}_\theta / \mathcal{T}_\theta$ -terms. Via the restriction and Coleman maps, we have an injection

$$H_{\text{Iw}}^1(\mathcal{O}_\infty, \tilde{\mathcal{T}}_\theta / \mathcal{T}_\theta(1)) \hookrightarrow X^{-1} \Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{M}_\theta / \mathfrak{S}_\theta \cong \Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta,$$

the latter isomorphism using [Oh2, Proposition 3.1.2] and Hypothesis 2.4.3d (though in the case said hypothesis fails, we have  $(\Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta)^2$  instead, and the result is the same). Clearly the latter module is  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta$ -torsion free. Also, Kummer theory provides us with the isomorphism in

$$H^1(\mathcal{O}, \tilde{\mathcal{T}}_\theta / \mathcal{T}_\theta(1)) \cong \bigoplus_{\ell | Np} (\tilde{\mathcal{T}}_\theta / \mathcal{T}_\theta)^\Delta \hookrightarrow \bigoplus_{\ell | Np} \Lambda_\theta,$$

the injection being a consequence of Theorem 2.3.9, [Oh2, Proposition 3.1.2], and Hypothesis 2.4.3d (the latter again being unnecessary for the result) and again the latter module is  $\Lambda_\theta$ -torsion free.  $\square$

**Lemma 5.1.2.** *Multiplication by  $1 - U_p$  is an injective endomorphism of  $H^1(\mathcal{O}, \mathcal{T}_\theta(1))$  and of  $H^1(\mathbb{Q}_p, \mathcal{T}_{\text{quo}}(1))$ .*

*Proof.* Since multiplication by  $1 - U_p$  is injective on  $\mathcal{T}_\theta$ , showing that  $1 - U_p$  is injective on  $H^1(\mathcal{O}, \mathcal{T}_\theta(1))$  amounts to showing that the Tate twist of  $\mathcal{T}_\theta / (U_p - 1)\mathcal{T}_\theta$  has trivial  $G_{\mathbb{Q}}$ -invariants. Note that the  $G_{\mathbb{Q}_p}$ -action on  $\mathcal{T}_{\text{quo}}$  is unramified, and therefore, the action of  $G_{\mathbb{Q}_p^{\text{ur}}}$  on  $\mathcal{T}_{\text{quo}}(1)$  is given by multiplication by the cyclotomic character. Therefore, we have

$$H^0(\mathbb{Q}_p, (\mathcal{T}_{\text{quo}} / (U_p - 1)\mathcal{T}_{\text{quo}})(1)) = 0$$

and the statement for  $H^1(\mathbb{Q}_p, \mathcal{T}_{\text{quo}}(1))$ . Since  $\mathcal{T}_\theta = \mathcal{T}_{\text{sub}} \oplus \mathcal{T}_{\text{quo}}$  as  $\mathfrak{h}_\theta$ -modules, it therefore suffices to show that no nontrivial element of  $(\mathcal{T}_{\text{sub}} / (U_p - 1)\mathcal{T}_{\text{sub}})(1)$  is fixed by  $G_{\mathbb{Q}}$  in  $(\mathcal{T}_\theta / (U_p - 1)\mathcal{T}_\theta)(1)$ .

Now,  $\mathcal{T}_{\text{sub}}$  is isomorphic to  $\mathfrak{h}_\theta$  as an  $\mathfrak{h}_\theta$ -module, and  $\mathcal{T}_{\text{sub}} / I\mathcal{T}_{\text{sub}}$  is isomorphic to the  $\mathfrak{h}_\theta[G_{\mathbb{Q}}]$ -quotient  $Q$  of  $\mathcal{T}_\theta / I\mathcal{T}_\theta$ . For  $\mathfrak{m} = I + (p, X)\mathfrak{h}$ , we have

$$(\mathcal{T}_{\text{sub}} / \mathfrak{m}\mathcal{T}_{\text{sub}})(1) \cong (\Lambda_\theta^1 / (p, X))(2) \cong (R_\theta / pR_\theta)(2)$$

as  $G_{\mathbb{Q}}$ -modules (where  $G_{\mathbb{Q}}$  acts on  $R_{\theta}$  through  $\theta^{-1}$ ), so has no fixed elements since  $\theta \neq \omega^2$ . If  $x \in (\mathcal{T}_{\text{sub}}/(U_p - 1)\mathcal{T}_{\text{sub}})(1)$  is nonzero and fixed by  $G_{\mathbb{Q}}$  inside  $(\mathcal{T}_{\theta}/(U_p - 1)\mathcal{T}_{\theta})(1)$ , then it is also fixed in  $x\mathcal{T}_{\text{sub}}/(x\mathfrak{m} + (U_p - 1)\mathcal{T}_{\text{sub}})(1)$  by the maximality of  $\mathfrak{m}\mathfrak{h}_{\theta}$ . This is isomorphic to a nonzero quotient of  $\mathcal{T}_{\text{sub}}/\mathfrak{m}\mathcal{T}_{\text{sub}}(1)$  under multiplication by  $x$ , so it has no fixed elements, which contradicts  $x \neq 0$ .  $\square$

**Lemma 5.1.3.** *For primes  $\ell \mid N$ , the groups  $H^1(\mathbb{Q}_{\ell}, \mathcal{T}_{\theta}(1))$  and  $H^1(\mathbb{Q}_{\ell}, \Lambda^1 \hat{\otimes}_{\mathbb{Z}_p} \mathcal{T}_{\theta}(1))$  are trivial.*

*Proof.* Fukaya and Kato showed that the inflation map

$$H^1(\mathbb{F}_{\ell}, H^0(\mathbb{Q}_{\ell}^{\text{ur}}, \mathcal{T}_{\theta}(1))) \rightarrow H^1(\mathbb{Q}_{\ell}, \mathcal{T}_{\theta}(1))$$

is an isomorphism [FK, 9.5.2], and their argument works with the degree  $p^f$  (unramified) extensions  $\mathbb{Q}_{\ell, r}$  and  $\mathbb{F}_{\ell, r}$  replacing  $\mathbb{Q}_{\ell}$  and  $\mathbb{F}_{\ell}$ , respectively. Note that  $\varprojlim_r H^1(\mathbb{F}_{\ell, r}, M)$  is trivial for any  $\mathbb{Z}_p[[G_{\mathbb{F}_{\ell}}]]$ -module  $M$  by Shapiro's lemma, so  $H^1(\mathbb{Q}_{\ell}, \Lambda^1 \hat{\otimes}_{\mathbb{Z}_p} \mathcal{T}_{\theta}(1)) = 0$ .

We have an exact sequence of  $\mathbb{Z}_p[G_{\mathbb{Q}_{\ell}}]$ -modules

$$0 \rightarrow \mathcal{T}_{\ell, \text{sub}} \rightarrow \mathcal{T}_{\theta} \rightarrow \mathcal{T}_{\ell, \text{quo}} \rightarrow 0,$$

where  $\mathcal{T}_{\ell, \text{sub}}$  and  $\mathcal{T}_{\ell, \text{quo}}$  have rank 1 over  $\mathfrak{h}_{\theta}$ , and the quotient  $\mathcal{T}_{\ell, \text{quo}}$  has an unramified action of  $G_{\mathbb{Q}_{\ell}}$ . Inertia at  $\ell$  acts on  $\mathcal{T}_{\ell, \text{sub}}$  by the restriction of the character  $\theta^{-1}$  that is primitive at  $\ell$ , so  $H^0(\mathbb{Q}_{\ell}^{\text{ur}}, \mathcal{T}_{\ell, \text{sub}}(1))$  is trivial. It thus remains only to show that  $H^1(\mathbb{F}_{\ell}, \mathcal{T}_{\ell, \text{quo}}(1))$  is trivial. As  $U_{\ell}$  acts  $\mathcal{T}_{\ell, \text{quo}}$  as a geometric Frobenius  $\Phi_{\ell}$  with eigenvalues congruent to 1 modulo  $I$ , the  $G_{\mathbb{F}_{\ell}}$ -action on  $\mathcal{T}_{\text{quo}}$  becomes trivial upon restriction to the Galois group of the unramified  $\mathbb{Z}_p$ -extension  $\mathbb{F}_{\ell, \infty}$  of  $\mathbb{F}_{\ell}$ . Since  $G_{\mathbb{F}_{\ell, \infty}}$  has no nontrivial  $p$ -quotient, inflation provides an isomorphism

$$H^1(\text{Gal}(\mathbb{F}_{\ell, \infty}/\mathbb{F}_{\ell}), H^0(\mathbb{F}_{\ell, \infty}, \mathcal{T}_{\ell, \text{quo}}(1))) \xrightarrow{\sim} H^1(\mathbb{F}_{\ell}, \mathcal{T}_{\ell, \text{quo}}(1)).$$

As  $\mathbb{F}_{\ell, \infty}$  does not contain a primitive  $p$ th root of unity, the group

$$H^0(\mathbb{F}_{\ell, \infty}, \mathcal{T}_{\ell, \text{quo}}(1)) \cong (\mathcal{T}_{\ell, \text{quo}} \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p(1))^{G_{\mathbb{F}_{\ell, \infty}}}$$

is zero. It follows that  $H^1(\mathbb{F}_{\ell}, H^0(\mathbb{Q}_{\ell}^{\text{ur}}, \mathcal{T}_{\theta}(1)))$  is trivial, as required.  $\square$

**Lemma 5.1.4.** *Under the identification of  $H_c^2(\mathcal{O}, P(1))$  with  $P$  of Lemma 3.1.16, the canonical map  $H^1(\mathbb{Q}_p, P(1)) \rightarrow H_c^2(\mathcal{O}, P(1))$  agrees with  $-\text{Col}_p^{\flat}$ .*

*Proof.* Consider the square

$$\begin{array}{ccc} H^1(\mathbb{Q}_p, P(1)) & \xrightarrow{\partial_p} & H^2(\mathbb{Q}_p, P(1)) \\ \downarrow & & \downarrow \\ H_c^2(\mathcal{O}, P(1)) & \xrightarrow{\partial_p} & H_c^3(\mathcal{O}, P(1)), \end{array}$$

which anticommutes by Lemma 3.1.15. The identifications of  $H^2(\mathbb{Q}_p, P(1))$  and  $H_c^3(\mathcal{O}, P(1))$  with  $P$  by invariant maps agree, and by Proposition 3.3.5 for  $\alpha = 1$ , the latter identification agrees with the identification of  $H_c^2(\mathcal{O}, P(1))$  with  $P$  via  $\partial_p$ . On the other hand, the composite map  $\text{inv} \circ \partial_p: H^1(\mathbb{Q}_p, P(1)) \rightarrow P$  equals  $\text{Col}_p^b$  by Lemma 4.1.7. The result follows.  $\square$

**Lemma 5.1.5.** *The exact sequences*

$$0 \rightarrow H^1(\mathcal{O}, P(1)) \rightarrow H^1(\mathcal{O}, T(1)) \rightarrow H^1(\mathcal{O}, Q(1)) \rightarrow 0$$

and

$$0 \rightarrow H^1(\mathcal{O}, P(1)) \rightarrow \bigoplus_{\ell|Np} H^1(\mathbb{Q}_\ell, P(1)) \rightarrow H_c^2(\mathcal{O}, P(1)) \rightarrow 0$$

are canonically split, compatibly with the map from the former sequence to the latter.

*Proof.* Since the  $G_{\mathbb{Q}}$ -action on  $P$  is trivial, we have isomorphisms

$$H^1(\mathbb{Q}_\ell, P(1)) \cong \varprojlim_r \mathbb{Q}_\ell^\times / \mathbb{Q}_\ell^{\times p^r} \hat{\otimes}_{\mathbb{Z}_p} P$$

for every  $\ell$  dividing  $Np$ . The  $\ell$ -adic valuation then induces a map from this group to  $P$  which is an isomorphism if  $\ell \neq p$  and otherwise induces a splitting of

$$0 \rightarrow H^1(\mathbb{Z}[\frac{1}{p}], P(1)) \rightarrow H^1(\mathbb{Q}_p, P(1)) \rightarrow H_c^2(\mathbb{Z}[\frac{1}{p}], P(1)) \rightarrow 0.$$

Noting that  $H_c^2(\mathbb{Z}[\frac{1}{p}], P(1))$  and  $H_c^2(\mathcal{O}, P(1))$  are canonically isomorphic, the sum of the  $\ell$ -adic valuation maps then gives the desired splitting of the injection in the second sequence. The splitting of the injection in the first sequence is given by the composition

$$H^1(\mathcal{O}, T(1)) \rightarrow \bigoplus_{\ell|Np} H^1(\mathbb{Q}_\ell, P(1)) \rightarrow H^1(\mathcal{O}, P(1)),$$

where the second map is the splitting of the second sequence. The final statement follows.  $\square$

We can now slightly refine the left-hand square in Theorem 3.1.17.

**Corollary 5.1.6.** *The square*

$$\begin{array}{ccc} H^1(\mathcal{O}, T(1)) & \longrightarrow & H^1(\mathcal{O}, Q(1)) \\ \downarrow & & \downarrow -\Theta \\ H^1(\mathbb{Q}_p, P(1)) & \longrightarrow & H_c^2(\mathcal{O}, P(1)) \end{array}$$

is commutative.

*Proof.* Both compositions are clearly trivial on elements of  $H^1(\mathcal{O}, P(1))$  inside  $H^1(\mathcal{O}, T(1))$ . On the other hand, Lemma 5.1.5 tells us that the composition

$$H^1(\mathcal{O}, Q(1)) \rightarrow H^1(\mathcal{O}, T(1)) \rightarrow \bigoplus_{\ell|Np} H^1(\mathbb{Q}_\ell, P(1))$$

takes image in the image of  $H_c^2(\mathcal{O}, P(1))$  (using the splittings induced by said lemma). This image is contained in  $H^1(\mathbb{Q}_p, P(1))$  inside the direct sum as the kernel of the  $p$ -adic valuation map  $H^1(\mathbb{Q}_p, P(1)) \rightarrow P$ . The result then follows from the commutativity of the left-hand square in Theorem 3.1.17.  $\square$

## 5.2 Modular symbols

We very briefly review modular symbols and Manin symbols.

**Definition 5.2.1.** For  $r \geq 1$  and cusps  $\alpha$  and  $\beta$  on  $X_1(Np^r)(\mathbb{C})$ , the *modular symbol*

$$\{\alpha \rightarrow \beta\}_r \in H_1(X_1(Np^r)(\mathbb{C}), \{\text{cusps}\}, \mathbb{Z}_p)$$

is the class of the geodesic from  $\alpha$  to  $\beta$  on  $X_1(Np^r)(\mathbb{C})$ .

**Definition 5.2.2.** For  $r \geq 1$  and  $u, v \in \mathbb{Z}/Np^r\mathbb{Z}$  with  $(u, v) = (1)$ , the *Manin symbol* of level  $Np^r$  attached to  $(u, v)$  is defined as

$$[u : v]_r = \left\{ \frac{-d}{bNp^r} \rightarrow \frac{-c}{aNp^r} \right\}_r = w_r \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \{0 \rightarrow \infty\}_r$$

for  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$  with  $u = c \pmod{Np^r}$  and  $v = d \pmod{Np^r}$ , where  $w_r$  is the Atkin-Lehner involution.

*Remark 5.2.3.* The Manin symbols of level  $Np^r$  generate  $H_1(X_1(Np^r)(\mathbb{C}), \{\text{cusps}\}, \mathbb{Z}_p)$ , and the relations

$$[u : v]_r = -[-v : u]_r = [u : u + v]_r + [u + v : v]_r$$

provide a presentation of said group.

**Definition 5.2.4.** For  $r \geq 1$ ,  $u, v \in \mathbb{Z}/Np^r\mathbb{Z}$  with  $(u, v) = (1)$ , and integers  $c, d > 1$  with  $(c, 6Np) = (d, 6Np) = 1$ , we have the  $(c, d)$ -symbol

$${}_{c,d}[u : v]_r = c^2 d^2 [u : v]_r - c^2 [u : dv]_r - d^2 [cu : v]_r + [cu : dv]_r.$$

The quotient maps  $X_1(Np^{r+1}) \rightarrow X_1(Np^r)$  take  $[pu : v]_{r+1}$  to  $[u : v]_r$  for  $u, v \in \mathbb{Z}/Np^{r+1}\mathbb{Z}$  with  $(u, v) = (1)$ , and similarly for the  $(c, d)$ -symbols. Let  $[u : v]_{r,\theta}^+$  denote the image of  $[u : v]_r$  in the  $\theta$ -eigenspace of the Eisenstein component of  $H_1(X_1(Np^r)(\mathbb{C}), \{\text{cusps}\}, \mathbb{Z}_p)^+$ .

**Definition 5.2.5.** For integers  $u$  and  $v$  with  $p \nmid v$  and  $(u, v, N) = 1$ , let

$$(u : v)_\theta = ([p^{r-1}u : v]_{r,\theta}^+)_r \in \mathcal{M}_\theta.$$

Note that  $(u : v)_\theta$  depends upon  $u$  only modulo  $Np$ . By [FK, 3.2.5], the elements  $(u : v)_\theta$  generate  $\mathcal{M}_\theta$ , and under Hypothesis 2.4.3, the group  $\mathcal{S}_\theta$  is generated by the symbols  $(u : v)_\theta$  with  $u \not\equiv 0 \pmod{Np}$  by [FK, 6.2.6].

**Definition 5.2.6.** Let

$$\Pi = \{(c, d, u, v) \in \mathbb{Z}^4 \mid c, d > 1, (c, 6Np) = (d, 6Np) = (u, v, N) = (v, p) = 1\},$$

and let

$$\Pi_0 = \{(c, d, u, v) \in \Pi \mid u \not\equiv 0 \pmod{Np}\}.$$

We define symbols attached to elements of these sets.

**Definition 5.2.7.** For  $(c, d, u, v) \in \Pi$ , let

$${}_{c,d}(u : v)_\theta = c^2 d^2 (u : v)_\theta - d^2 (cu : v)_\theta - c^2 (u : dv)_\theta + (cu : dv)_\theta \in \mathcal{M}_\theta,$$

and define  ${}_{c,d}(u : v]_\theta \in \Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathcal{M}_\theta$  by

$${}_{c,d}(u : v]_\theta = c^2 d^2 \otimes (u : v)_\theta - d^2 \kappa(c) \otimes (cu : v)_\theta - c^2 \kappa(d) \otimes (u : dv)_\theta + \kappa(cd) \otimes (cu : dv)_\theta,$$

where  $\kappa : \mathbb{Z}_p^\times \rightarrow \Lambda$  sends a unit to the group element of its projection to  $1 + p\mathbb{Z}_p$ .

### 5.3 Zeta elements

We first very briefly recall the Kato-Beilinson elements (or zeta elements) of [FK, Section 2]. We then, in the form we shall require, slightly refine the resulting maps of Fukaya and Kato [FK, Section 3] and describe the properties of them that we need.

The following definition is from [FK, 2.4.2].



**Definition 5.3.1.** For  $r, s \geq 0$  and  $u, v \in \mathbb{Z}$  with  $(u, v, Np) = (1)$ , and supposing that  $u, v \not\equiv 0 \pmod{Np^r}$  if  $s = 0$ , we define  ${}_{c,d}z_{r,s}(u : v)$  to be the image under the norm and Hochschild-Serre maps

$$H^2(Y(p^s, Np^{r+s})/\mathbb{Z}[\frac{1}{Np}], \mathbb{Z}_p(2)) \rightarrow H^2(Y_1(Np^r)/\mathcal{O}_s, \mathbb{Z}_p(2)) \rightarrow H^1(\mathcal{O}_s, H^1(Y_1(Np^r)/\overline{\mathbb{Q}}, \mathbb{Z}_p(2)))$$

of the cup product  $cg_{\frac{a}{p^s}, \frac{c}{Np^{r+s}}} \cup dg_{\frac{b}{p^s}, \frac{d}{Np^{r+s}}}$  of Siegel units on  $Y(p^s, Np^{r+s})/\mathbb{Z}[\frac{1}{Np}]$ , where  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$  with  $u = c \pmod{Np^r}$  and  $v = d \pmod{Np^r}$ .

*Remark 5.3.2.* As a consequence of [FK, 2.4.4, 3.1.9], the elements  ${}_{c,d}z_{r,s}(u : v)$  are for  $r, s \geq 1$  compatible with the maps induced by quotients of modular curves and corestriction maps for the ring extensions. Moreover, the corestriction map

$$H^1(\mathcal{O}_s, H^1(Y_1(Np^r)/\overline{\mathbb{Q}}, \mathbb{Z}_p(2))) \rightarrow H^1(\mathcal{O}, H^1(Y_1(Np^r)/\overline{\mathbb{Q}}, \mathbb{Z}_p(2)))$$

takes  ${}_{c,d}z_{r,s}(u : v)$  to  $(1 - U_p)_{c,d}z_{r,0}(u : v)$  if  $(c, d, u, v) \in \Pi_0$ .

Let us use  ${}_{c,d}z_{r,s}(u : v)_\theta$  to denote the projection of  ${}_{c,d}z_{r,s}(u : v)$  to the Eisenstein component for  $\theta$ .

**Definition 5.3.3.** For  $(c, d, u, v) \in \Pi$ , we set

$${}_{c,d}z(u : v)_\theta = ({}_{c,d}z_{r,s}(u : v)_\theta)_{r,s \geq 1} \in H^1_{\mathrm{Iw}}(\mathcal{O}_\infty, \tilde{\mathcal{T}}_\theta(1))$$

and for  $(c, d, u, v) \in \Pi_0$ , we set

$${}_{c,d}z^\sharp(u : v)_\theta = ({}_{c,d}z_{r,0}(u : v)_\theta)_{r \geq 1} \in H^1(\mathcal{O}, \tilde{\mathcal{T}}_\theta(1)).$$

By Remark 5.3.2, the corestriction of  ${}_{c,d}z(u : v)_\theta$  to  $H^1(\mathcal{O}, \tilde{\mathcal{T}}_\theta(1))$  is  $(1 - U_p)z^\sharp(u : v)_\theta$  for  $(c, d, u, v) \in \Pi_0$

**Definition 5.3.4.** Let  $\mathcal{Z}$  denote the unique element of  $X^{-1}\mathbb{Z}_p[[X]]$  such that

$$\mathcal{Z}(t^s - 1) = \zeta_p(1 - s)$$

for all  $s \in \mathbb{Z}_p$ , where  $\zeta_p$  denotes the  $p$ -adic Riemann zeta function.

The following result, constructing a *zeta map*, is a refinement of a result of Fukaya and Kato [FK, 3.3.3]. It is in essence a consequence of [FKS2, Theorem 3.15].

**Theorem 5.3.5.** *There exists a  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{h}_\theta$ -module homomorphism*

$$z: \Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathcal{S}_\theta \rightarrow H_{\text{Iw}}^1(\mathcal{O}_\infty, \mathcal{T}_\theta(1))$$

*such that its composition with the injective map*

$$H_{\text{Iw}}^1(\mathcal{O}_\infty, \mathcal{T}_\theta(1)) \rightarrow H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, \mathcal{T}_\theta(1))$$

*equals the map  $z_{\text{quo}}$  of Definition 4.3.2 for an element  $\alpha \in \Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{h}_\theta$  with image  $X \mathcal{Z} \tilde{\xi}_1 \in \Lambda \hat{\otimes}_{\mathbb{Z}_p} (\mathfrak{h}/I)_\theta$ .*

*Proof.* In [FKS2, Theorem 3.15], we show the existence of a map

$$\tilde{z}: \Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathcal{M}_\theta \rightarrow H_{\text{Iw}}^1(\mathcal{O}_\infty, \tilde{\mathcal{T}}_\theta(1))$$

of  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{H}_\theta$ -modules that satisfies

$$\tilde{z}_{(c,d}(u : v)_\theta) = -_{c,d} z(u : v)_\theta \otimes 1$$

for all  $(c, d, u, v) \in \Pi$ .

By composition with restriction, we obtain a map

$$\tilde{z}_{\text{quo}}: \Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathcal{M}_\theta \rightarrow H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, \tilde{\mathcal{T}}_{\text{quo}}(1)).$$

Via the fixed isomorphism  $\mathcal{M}_\theta \cong \mathfrak{M}_\theta$  and the canonical isomorphism

$$\text{End}_{\Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{H}_\theta}(\Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathcal{M}_\theta) \xrightarrow{\sim} \Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{H}_\theta,$$

the map  $\text{Col} \circ \tilde{z}_{\text{quo}}$  is given by multiplication by an element  $\beta$  in  $X^{-1} \Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{H}_\theta$ . This map induces an endomorphism of  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathcal{S}_\theta$  by [FK, 4.4.3], and the resulting map on the latter module is given by multiplication by the image  $\alpha \in \Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{h}_\theta$  of  $\beta$  in  $X^{-1} \Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{h}_\theta$ . The element  $\alpha$  reduces to  $X \mathcal{Z} \tilde{\xi}_1 \in \Lambda \hat{\otimes}_{\mathbb{Z}_p} (\mathfrak{h}/I)_\theta$  by [FK, 8.1.2(1)]. Note that the congruence  $X \mathcal{Z} \equiv 1 \pmod{X}$  implies that the image of  $\alpha$  in  $(\mathfrak{h}/I)_\theta^\dagger$  agrees with that of  $\tilde{\xi}_1$ , so  $\alpha$  has the property of Definition 4.3.1. Thus,  $\tilde{z}_{\text{quo}}$  restricted to  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathcal{S}_\theta$  equals  $z_{\text{quo}}$  of Definition 4.3.2 for this value of  $\alpha$ .

The composition

$$H_{\text{Iw}}^1(\mathcal{O}_\infty, \tilde{\mathcal{T}}_\theta(1)) \rightarrow H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, \tilde{\mathcal{T}}_\theta(1)) \rightarrow X^{-1} \Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{M}_\theta$$

is injective by [FK, 3.1.4 and 4.2.10] and Lemma 5.1.1. We must prove the claim that the restriction of  $\tilde{z}$  to  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathcal{S}_\theta$  takes values in  $H_{\text{Iw}}^1(\mathcal{O}_\infty, \mathcal{T}_\theta(1))$ . By what we have shown above,  $\tilde{z}$  carries  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathcal{S}_\theta$  to the kernel of

$$H_{\text{Iw}}^1(\mathcal{O}_\infty, \tilde{\mathcal{T}}_\theta / \mathcal{T}_\theta(1)) \rightarrow H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, \tilde{\mathcal{T}}_{\text{quo}} / \mathcal{T}_{\text{quo}}(1)) \hookrightarrow X^{-1} \Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{M}_\theta / \mathfrak{S}_\theta.$$

Note that  $\tilde{\mathcal{T}}_\theta/\mathcal{T}_\theta \cong (\mathfrak{H}/I)_\theta^1(1)$  as  $\mathfrak{h}[G_\mathbb{Q}]$ -modules, which in turn are canonically isomorphic to  $\tilde{\mathcal{T}}_{\text{quo}}/\mathcal{T}_{\text{quo}}$  as  $\mathfrak{h}[G_{\mathbb{Q}_p}]$ -modules. This kernel is then trivial as a consequence of weak Leopoldt. By the exactness of the first sequence in Lemma 5.1.1, we have the claim.  $\square$

From now on, we take  $\alpha = \alpha_\theta$  to be as given in Theorem 5.3.5. We prove the following slight refinement of [FK, 3.3.9] on a zeta map at the level of  $\mathbb{Q}$  as a consequence of [FKS2, Theorem 3.17].

**Theorem 5.3.6.** *There exists a unique map*

$$z^\sharp: \mathcal{S}_\theta \rightarrow H^1(\mathcal{O}, \mathcal{T}_\theta(1))$$

of  $\mathfrak{h}_\theta$ -modules with the property that for the map  $\text{cor}: H_{\text{Iw}}^1(\mathcal{O}_\infty, \mathcal{T}_\theta(1)) \rightarrow H^1(\mathcal{O}, \mathcal{T}_\theta(1))$  induced by corestriction we have

$$\text{cor} \circ z = (1 - U_p)z^\sharp \circ \text{ev}_0.$$

The composition of  $z^\sharp$  with

$$H^1(\mathcal{O}, \mathcal{T}_\theta(1)) \rightarrow H^1(\mathbb{Q}_p, \mathcal{T}_\theta(1))$$

equals the map  $z_{\text{quo}}^\sharp$  of Proposition 4.3.3 for  $\alpha$  as in Theorem 5.3.5.

*Proof.* In [FKS2, Theorem 3.17] (noting [FK, 3.3.14]), we prove (using Lemma 5.1.1 of this paper) the existence of an  $\mathfrak{H}_\theta$ -module homomorphism

$$z^\sharp: \mathcal{S}_\theta \rightarrow H^1(\mathcal{O}, \mathcal{T}_\theta(1))$$

with the property that

$$z^\sharp(c, d(u : v)_\theta) = -_{c, d} z^\sharp(u : v)_\theta \tag{5.1}$$

for all  $(c, d, u, v) \in \Pi_0$ .

The comparison with  $z$  is [FK, 3.3.9(ii)], the uniqueness being Lemma 5.1.2. The comparison with  $z_{\text{quo}}^\sharp$  follows from Proposition 4.3.3, the comparison with  $z$ , and Theorem 5.3.5.  $\square$

Fukaya and Kato prove the following in [FK, 5.2.10-11 and 9.2.1]. We sketch their proof primarily to make clear how to obtain the sign in its comparison. That is, there are two sign differences from their proof which effectively cancel each other, and the sign of the second map in the composition in its statement is the opposite of that of [FK, 6.3.9].

**Theorem 5.3.7** (Fukaya-Kato). *The composition of  $z^\sharp$  with the maps*

$$H^1(\mathcal{O}, \mathcal{T}_\theta(1)) \rightarrow H^1(\mathcal{O}, Q(1)) \xrightarrow{\sim} Y$$

given by Proposition 2.4.9 and Lemma 3.1.4 equals  $-\bar{\omega}$ .

*Proof.* The results of Sections 5.1 and 5.2 of [FK] yield a map

$$\infty^* : H^1(\mathcal{O}, \mathcal{T}_\theta(1)) \xrightarrow{\sim} \varprojlim_r H^2(Y_1(Np^r)_{/\mathcal{O}}, \mathbb{Z}_p(2))_{\mathfrak{m}, \theta} \rightarrow Y,$$

given by composing the inverse of the Hochschild-Serre map with a specialization-at- $\infty$  map. It follows from [FK, 5.1.9] that this composition takes  ${}_{c,d}z^\sharp(u : v)_\theta$  to  $\overline{\omega}(c,d(u : v)_\theta)$ . (Note that the sign here is opposite to that in the proof of [FK, 5.2.11] in the current version, as an unexplained sign appears in the proof of Claim 1 therein.) By (5.1) and [FKS2, Proposition 3.1.6], the composition  $\infty^* \circ z^\sharp$  is then  $-\overline{\omega}$ .

By [FK, 9.2.5], the map  $\infty^*$  agrees with the connecting map

$$H^1(\mathcal{O}, \mathcal{T}_\theta(1)) \rightarrow H^2(\mathcal{O}, \Lambda_\theta^1(2))$$

in the long exact sequence associated to the Tate twist of the short exact sequence

$$0 \rightarrow \Lambda_\theta^1(1) \rightarrow \tilde{\mathcal{T}}_{c,\theta} \rightarrow \mathcal{T}_\theta \rightarrow 0,$$

where  $1 \in \Lambda_\theta^1(1)$  corresponds to the cusp at  $\infty$ . It then suffices to show that there is a commutative diagram of continuous  $G_{\mathbb{Q}}$ -module homomorphisms

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Lambda_\theta^1(1) & \longrightarrow & \tilde{\mathcal{T}}_{c,\theta} & \longrightarrow & \mathcal{T}_\theta \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Lambda_\theta^1(1) & \xrightarrow{\xi} & \Lambda_\theta^1(1) & \longrightarrow & \mathcal{Q} \longrightarrow 0, \end{array} \quad (5.2)$$

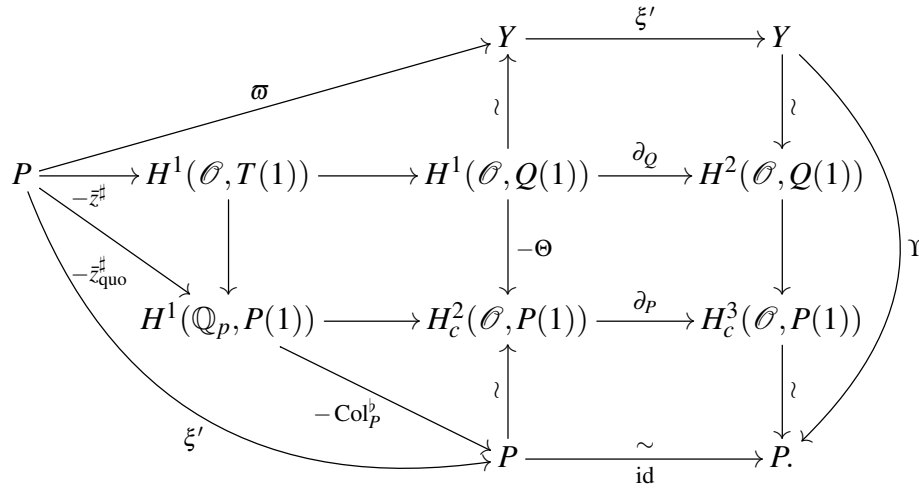
where the right-hand vertical map is the surjection of Proposition 2.4.9 given by  $v \mapsto \langle \xi e_\infty, v \rangle$  modulo  $\xi$ .

Consider the element  $g \in \tilde{\mathcal{T}}_\theta$  given by the  $\theta$ -projection of the compatible system of Siegel units  $(g_{0, \frac{1}{Np^r}})_r$  and Kummer theory. The boundary map  $\tilde{\mathcal{T}}_\theta \rightarrow \Lambda_\theta$  at 0-cusps of [FK, 6.2.5] carries  $g$  to  $-\xi$ , the equivariant sum of its orders of vanishing at the 0-cusps; for this, see [MW, I.6 (3) and IV.2 (1)], as well as the equality of the first and last terms in (2.1). (Note that we obtain here the opposite sign to [FK, 6.2.13], which leads us to replace  $g$  with  $g^{-1}$  in the pairing map below, without further effect.) Much as in [FK, 9.2.3], the desired commutative diagram (5.2) is given by taking the center vertical map to be the pairing map  $w \mapsto \langle g^{-1}, w \rangle$ , which is  $G_{\mathbb{Q}}$ -equivariant as  $g$  is  $G_{\mathbb{Q}}$ -fixed. For the right-hand square, the commutativity is just as in [FK, 9.2.3], but note that we have changed the side on which we are pairing with  $\xi e_\infty$  in Proposition 2.4.9. The left-hand square commutes (instead of anticommutes, as suggested by the proof of [FK, 9.2.3]) as a consequence of the first equality in (2.1), though here we use Ohta's twisted Poincaré pairing (2.3), which is a system of compatible pairings, each involving an Atkin-Lehner involution that takes  $\infty$ -cusps to 0-cusps (cf. [FK, 1.6.2]).  $\square$

The main result [FK, 0.14] in the work of Fukaya and Kato states that  $\xi' \Upsilon \circ \varpi$  and  $\xi'$  induce the same endomorphism of  $P \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ . As  $P$  is not known to be  $p$ -torsion free, this is slightly weaker than equality as endomorphisms of  $P$ . With the results of [FKS2] in hand, it is now a relatively straightforward matter to show that the stronger statement holds by following the argument of [FK].

**Theorem 5.3.8** (Fukaya-Kato, Fukaya-Kato-S.). *One has  $\xi' \Upsilon \circ \varpi = \xi' \in \text{End}_{\mathfrak{h}_\theta}(P)$ .*

*Proof.* Theorems 3.1.17, 5.3.6 and 5.3.7, Proposition 4.3.3, Corollary 5.1.6, and Lemma 5.1.4 provide a commutative diagram



□

## 5.4 Refined global cohomology

We prove analogues for intermediate cohomology of earlier results on global cohomology. We begin with an extension of Lemma 5.1.4. Let use  $\overline{\text{Col}}_P^\dagger: H^1(\mathbb{Q}_p, P^\dagger(1)) \rightarrow P$  to denote the composition  $\psi \circ \text{Col}_P^\dagger$ .

**Lemma 5.4.1.** *Under the identification of  $H_c^2(\mathcal{O}, P^\dagger(1))$  with  $P$  of Lemma 3.1.16, the canonical map  $H^1(\mathbb{Q}_p, P^\dagger(1)) \rightarrow H_c^2(\mathcal{O}, P^\dagger(1))$  agrees with  $-\overline{\text{Col}}_P^\dagger$ .*

*Proof.* The anticommutativity of the square

$$\begin{array}{ccc} H^1(\mathbb{Q}_p, P^\dagger(1)) & \xrightarrow{\partial_P^\dagger} & H^2(\mathbb{Q}_p, P(1)) \\ \downarrow & & \downarrow \\ H_c^2(\mathcal{O}, P^\dagger(1)) & \xrightarrow{\partial_P^\dagger} & H_c^3(\mathcal{O}, P(1)), \end{array}$$

is proven by the analogous argument to Lemma 3.1.15, and the identifications of  $H^2(\mathbb{Q}_p, P(1))$  and  $H_c^3(\mathcal{O}, P(1))$  with  $P$  agree as before. By Proposition 3.3.5, the latter identification agrees via  $\partial_p^\dagger$  with the identification of  $H_c^2(\mathcal{O}, P^\dagger(1))$  with  $P$ . Finally,  $\overline{\text{Col}}_P^\dagger = \text{inv} \circ \partial_p^\dagger$  by the commutativity of (4.2).  $\square$

Next, we have an analogue of Lemma 5.1.5.

**Proposition 5.4.2.** *The exact sequences*

$$0 \rightarrow H^1(\mathcal{O}, P^\dagger(1)) \rightarrow H^1(\mathcal{O}, T^\dagger(1)) \rightarrow H^1(\mathcal{O}, Q^\dagger(1)) \rightarrow 0$$

and

$$0 \rightarrow H^1(\mathcal{O}, P^\dagger(1)) \rightarrow \bigoplus_{\ell|N_p} H^1(\mathbb{Q}_\ell, P^\dagger(1)) \rightarrow H_c^2(\mathcal{O}, P^\dagger(1)) \rightarrow 0$$

are canonically split, compatibly with the map from the former sequence to the latter. The splitting of the surjection in the latter sequence takes image in  $H^1(\mathbb{Q}_p, P^\dagger(1))$  inside the direct sum. Moreover, the splittings are compatible with the maps of these sequences to (via the quotient map  $P^\dagger \rightarrow P$ ) and from (via  $\alpha: P \rightarrow P^\dagger$ ) the corresponding split sequences of Lemma 5.1.5.

*Proof.* Since  $\text{Fr}_p$  acts trivially on  $P$ , the exact sequence

$$0 \rightarrow X^{-1}(\Lambda \hat{\otimes}_{\mathbb{Z}_p} P) / \alpha(\Lambda \hat{\otimes}_{\mathbb{Z}_p} P) \rightarrow H^1(\mathbb{Q}_p, P^\dagger(1)) \xrightarrow{\overline{\text{Col}}_P^\dagger} P \rightarrow 0$$

of Theorem 4.2.4 is canonically split, with the first term identified with  $H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, P(1)) / X\alpha$  and the third identified with  $H_{\text{Iw}}^2(\mathbb{Q}_{p,\infty}, P(1)) \xrightarrow{\sim} H^2(\mathbb{Q}_p, P^\dagger(1))$ . As the restriction map

$$H_{\text{Iw}}^1(\mathbb{Z}_\infty[\frac{1}{p}], P(1)) \rightarrow H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, P(1))$$

is an isomorphism and  $H^2(\mathcal{O}, P^\dagger(1)) = 0$ , the first term can be replaced by  $H^1(\mathbb{Z}[\frac{1}{p}], P^\dagger(1))$ , and the third term  $P$  can then be replaced by  $H_c^2(\mathcal{O}, P^\dagger(1))$ . (To obtain the usual Poitou-Tate sequence, the isomorphism of  $H_c^2(\mathcal{O}, P^\dagger(1))$  with  $P$  used here is the negative of our fixed identification by Lemma 5.4.1.) That is, the exact sequence

$$0 \rightarrow H^1(\mathbb{Z}[\frac{1}{p}], P^\dagger(1)) \rightarrow H^1(\mathbb{Q}_p, P^\dagger(1)) \rightarrow H_c^2(\mathcal{O}, P^\dagger(1)) \rightarrow 0$$

canonically split. We then take the splitting of the surjection in the second exact sequence in the statement to be given by the composition

$$H_c^2(\mathcal{O}, P^\dagger(1)) \rightarrow H^1(\mathbb{Q}_p, P^\dagger(1)) \rightarrow \bigoplus_{\ell|N_p} H^1(\mathbb{Q}_\ell, P^\dagger(1)),$$

where the latter map is the inclusion of the summand for  $\ell = p$ .

The composition

$$H^1(\mathcal{O}, T^\dagger(1)) \rightarrow \bigoplus_{\ell|Np} H^1(\mathbb{Q}_\ell, P^\dagger(1)) \rightarrow H^1(\mathcal{O}, P^\dagger(1))$$

then gives a canonical splitting of the first exact sequence which is compatible with the map between the two.

The final statement follows easily from the fact that the splitting of  $\overline{\text{Col}}^\dagger : H^1(\mathbb{Q}_p, P^\dagger(1)) \rightarrow P$  is given by the composition of the canonical splitting  $(\text{Col}_p^\flat)^{-1} : P \rightarrow H^1(\mathbb{Q}_p, P(1))$  of  $\text{Col}_p^\flat$  given by the valuation map, as in Remark 4.1.6, with the map induced by  $\alpha : P \rightarrow P^\dagger$ . The latter composition is the pushout map  $D(P) \rightarrow \mathfrak{C}^*(P)$  constructed in the proof of Theorem 4.2.4. For this, note that

$$\begin{array}{ccccc} H^1(\mathbb{Q}_p, P^\dagger(1)) & \longrightarrow & H_{\text{Iw}}^2(\mathbb{Q}_{p,\infty}, P(1)) & \xrightarrow{\text{inv}} & P \\ \downarrow & & \downarrow \alpha & & \downarrow \alpha(0) \\ H^1(\mathbb{Q}_p, P(1)) & \longrightarrow & H_{\text{Iw}}^2(\mathbb{Q}_{p,\infty}, P(1)) & \xrightarrow{\text{inv}} & P \\ \downarrow \alpha & & \parallel & & \parallel \\ H^1(\mathbb{Q}_p, P^\dagger(1)) & \longrightarrow & H_{\text{Iw}}^2(\mathbb{Q}_{p,\infty}, P(1)) & \xrightarrow{\text{inv}} & P \end{array}$$

commutes, with the horizontal compositions being  $\overline{\text{Col}}^\dagger$  in the first and last rows and  $\overline{\text{Col}}_p^\flat$  in the middle row, as seen in the commutative diagram (4.2).  $\square$

**Corollary 5.4.3.** *The square*

$$\begin{array}{ccc} H^1(\mathcal{O}, T^\dagger(1)) & \longrightarrow & H^1(\mathcal{O}, Q^\dagger(1)) \\ \downarrow & & \downarrow -\Theta^\dagger \\ H^1(\mathbb{Q}_p, P^\dagger(1)) & \longrightarrow & H_c^2(\mathcal{O}, P^\dagger(1)) \end{array}$$

*is commutative.*

*Proof.* Both compositions are trivial on the image of  $H^1(\mathcal{O}, P^\dagger(1))$  in  $H^1(\mathcal{O}, T^\dagger(1))$ . Proposition 5.4.2 implies that the composition (using the splittings of said proposition)

$$H^1(\mathcal{O}, Q^\dagger(1)) \rightarrow H^1(\mathcal{O}, T^\dagger(1)) \rightarrow \bigoplus_{\ell|Np} H^1(\mathbb{Q}_\ell, P^\dagger(1))$$

takes image in the image of  $H_c^2(\mathcal{O}, P^\dagger(1))$ . This image is contained in  $H^1(\mathbb{Q}_p, P^\dagger(1))$  by construction. The result then follows from the commutativity of the square in Proposition 3.3.9.  $\square$

## 5.5 Refined zeta maps

In this subsection, we show how the existence of a refined zeta map would imply Conjecture 3.1.9. In fact, we show that the question of the existence of a reduced such map is equivalent to the conjecture.

**Theorem 5.5.1.** *Conjecture 3.1.9 holds if and only if there exists a map  $\bar{z}^\dagger: \Lambda \otimes_{\mathbb{Z}_p} P \rightarrow H^1(\mathcal{O}, T^\dagger(1))$  of  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{h}_\theta/I$ -modules making the diagrams*

$$\begin{array}{ccc}
 \Lambda \otimes_{\mathbb{Z}_p} P & \xrightarrow{\bar{z}^\dagger} & H^1(\mathcal{O}, T^\dagger(1)) \\
 \downarrow & & \downarrow \\
 P & \xrightarrow{\bar{z}^\sharp} & H^1(\mathcal{O}, T(1))
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 \Lambda \otimes_{\mathbb{Z}_p} P & \xrightarrow{\bar{z}^\dagger} & H^1(\mathcal{O}, T^\dagger(1)) \\
 \downarrow & & \downarrow \\
 P & \xrightarrow{\bar{z}^\dagger_{\text{quo}}} & H^1(\mathbb{Q}_p, P^\dagger(1))
 \end{array}$$

commute, where the vertical maps in the first diagram are induced by the augmentation on  $\Lambda$ .

*Proof.* By Remark 3.1.10, Conjecture 3.1.9 holds if and only if  $\Upsilon \circ \varpi = 1$ . We use this form of the conjecture. If a map  $\bar{z}^\dagger$  as in the statement exists, then the composition of  $-\bar{z}^\dagger$  with the maps

$$H^1(\mathcal{O}, T^\dagger(1)) \rightarrow H^1(\mathcal{O}, Q^\dagger(1)) \rightarrow H^1(\mathcal{O}, Q(1)) \xrightarrow{\sim} Y$$

is necessarily  $\varpi$  (after application of  $\text{ev}_0$ ) by the commutativity of the first diagram and Theorem 5.3.7, and its further composition via  $-\Theta^\dagger$  to  $H_c^2(\mathcal{O}, P^\dagger(1))$  is necessarily  $\Upsilon \circ \varpi$  since we have seen that  $-\Theta^\dagger$  induces a map  $Y \rightarrow P$  that equals  $\Upsilon$  in Section 3.3. At the same time, by the commutativity of the second diagram, this composition agrees with the composition of  $-\bar{z}^\dagger_{\text{quo}} \circ \text{ev}_0$  with  $H^1(\mathbb{Q}_p, P^\dagger(1)) \rightarrow H_c^2(\mathcal{O}, P^\dagger(1)) \cong P$ , which is 1 by Lemma 5.4.1 and Proposition 4.3.4. Thus  $\Upsilon \circ \varpi = 1$ .

Conversely, suppose that  $\Upsilon \circ \varpi = 1$ . Using the isomorphism

$$H^1(\mathcal{O}, T^\dagger(1)) \cong H^1(\mathcal{O}, P^\dagger(1)) \oplus H^1(\mathcal{O}, Q^\dagger(1))$$

given by the splitting of Proposition 5.4.2, we may define the projections of  $\bar{z}^\dagger$  to these components separately: let us label them  $\bar{z}_P^\dagger$  and  $\bar{z}_Q^\dagger$ , respectively. We will similarly write  $\bar{z}^\sharp$  as the sum of its components  $\bar{z}_P^\sharp$  and  $\bar{z}_Q^\sharp$  corresponding to projection to  $H^1(\mathcal{O}, P(1))$  and  $H^1(\mathcal{O}, Q(1))$ ,

Recall that we have

$$H^1(\mathcal{O}, Q^\dagger(1)) \cong (\Lambda \otimes_{\mathbb{Z}_p} Y) / \xi(\Lambda \otimes_{\mathbb{Z}_p} Y)$$

where  $f \in \Lambda \hat{\otimes}_{\mathbb{Z}_p} \Lambda_\theta$  acts as  $w(f)$  on  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} Y$ . Using this isomorphism to identify the two sides, we set

$$\bar{z}_Q^\dagger = -1 \otimes \varpi \bmod \xi(\Lambda \otimes_{\mathbb{Z}_p} Y).$$



Via the splitting of Lemma 5.1.5, we have an isomorphism

$$\bigoplus_{\ell|Np} H^1(\mathbb{Q}_\ell, P^\dagger(1)) \cong H^1(\mathcal{O}, P^\dagger(1)) \oplus H_c^2(\mathcal{O}, P^\dagger(1)),$$

and we let  $\bar{z}_p^\dagger$  be the projection of  $\bar{z}_{\text{quo}}^\dagger \circ \text{ev}_0$  to the first component.

We next check commutativity of the the first diagram. By Proposition 5.4.2, we may do this after projection to the summands corresponding to  $P$  and  $Q$ , respectively. For the  $P$ -components, note that  $\bar{z}_p^\dagger$  is the projection of  $\bar{z}_{\text{quo}}^\dagger \circ \text{ev}_0$  to  $H^1(\mathcal{O}, P^\dagger(1))$ . The composition of this map with the surjection to  $H^1(\mathcal{O}, P(1))$  is the projection of  $\bar{z}_{\text{quo}}^\sharp \circ \text{ev}_0$  to  $H^1(\mathcal{O}, P(1))$ . This equals  $\bar{z}_p^\sharp \circ \text{ev}_0$  in that the restriction of  $\bar{z}_p^\sharp$  to  $H^1(\mathbb{Q}_\ell, P(1))$  is trivial for primes  $\ell | N$ . That is,  $\bar{z}_p^\sharp$  is a reduction of  $z^\sharp: \mathcal{S}_\theta \rightarrow H^1(\mathcal{O}, \mathcal{T}_\theta(1))$ , and  $H^1(\mathbb{Q}_\ell, \mathcal{T}_\theta(1))$  is trivial by Lemma 5.1.3.

For the  $Q$ -components, we need only remark that the composition of  $\bar{z}_Q^\dagger = -1 \otimes \bar{\omega}$  with the map to  $H^1(\mathcal{O}, Q(1))$  is  $\bar{z}_Q^\sharp = -\bar{\omega}$  (see Proposition 3.3.4), so we see that the first diagram commutes on the summands corresponding to  $Q$ .

For the second diagram, we have by definition that  $\bar{z}_p^\dagger$  equals the projection of  $\bar{z}_{\text{quo}}^\dagger \circ \text{ev}_0$  to  $H^1(\mathcal{O}, P^\dagger(1))$ . The composition of  $-\bar{z}_Q^\dagger = 1 \otimes \bar{\omega}$  with  $-\Theta^\dagger: H^1(\mathcal{O}, Q^\dagger(1)) \rightarrow H_c^2(\mathcal{O}, P^\dagger(1))$ , the latter group being identified with  $P$ , factors through  $\Upsilon \circ \bar{\omega} = 1$ . As the composition of  $-\bar{z}_{\text{quo}}^\dagger$  with  $H^1(\mathbb{Q}_p, P^\dagger(1)) \rightarrow H_c^2(\mathcal{O}, P^\dagger(1))$  is also identified with 1, the commutativity holds.  $\square$

Note that the data of  $\bar{z}^\dagger$  is equivalent to the data of its restriction to an  $\mathfrak{h}_\theta/I$ -module homomorphism  $P \rightarrow H^1(\mathcal{O}, T^\dagger(1))$  sending  $x \in P$  to  $\bar{z}^\dagger(1 \otimes x)$  and fitting in the corresponding commutative diagrams arising from restriction to  $P$ .

The above discussion can be summarized by the diagram

$$\begin{array}{ccccccc}
P & \xrightarrow{\bar{\omega}} & Y & \xrightarrow{\text{id}} & Y & & \\
\uparrow \text{ev}_0 & & \uparrow & & \downarrow \wr & & \\
\Lambda \otimes_{\mathbb{Z}_p} P & \xrightarrow[-\bar{z}^\dagger]{} & H^1(\mathcal{O}, T^\dagger(1)) & \longrightarrow & H^1(\mathcal{O}, Q^\dagger(1)) & \xrightarrow{\partial_Q^\dagger} & H^2(\mathcal{O}, Q(1)) \\
\downarrow \text{ev}_0 & & \downarrow & & \downarrow -\Theta^\dagger & & \downarrow \\
P & \xrightarrow[-\bar{z}_{\text{quo}}^\dagger]{} & H^1(\mathbb{Q}_p, P^\dagger(1)) & \longrightarrow & H_c^2(\mathcal{O}, P^\dagger(1)) & \xrightarrow{\partial_P^\dagger} & H_c^3(\mathcal{O}, P(1)) \\
& & \downarrow & & \downarrow \wr & & \downarrow \wr \\
& & P & \xrightarrow{\text{id}} & P & & P
\end{array}$$

$\Upsilon$

which fully commutes if we know the existence of the conjectural map  $\bar{z}^\dagger$  in Theorem 5.5.1. The equality  $\Upsilon \circ \bar{\omega} = 1$  is then seen by tracing the outside of the diagram. This begs the

following question, which would be in analogy with the construction of  $z^\sharp$  by Fukaya and Kato were it to hold.

*Question 5.5.2.* Does there exist a  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{h}_\theta$ -module homomorphism

$$z^\dagger : \Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathcal{T}_\theta \rightarrow H^1(\mathcal{O}, \mathcal{T}_\theta^\dagger(1))$$

such that the diagram

$$\begin{array}{ccc} \Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathcal{T}_\theta & \xrightarrow{z^\dagger} & H^1(\mathcal{O}, \mathcal{T}_\theta^\dagger(1)) \\ \downarrow z & & \downarrow 1-U_p \\ H_{\text{Iw}}^1(\mathcal{O}_\infty, \mathcal{T}_\theta(1)) & \longrightarrow & H^1(\mathcal{O}, \mathcal{T}_\theta^\dagger(1)) \end{array}$$

commutes?

Such a map  $z^\dagger$  is uniquely determined if it exists by the following lemma.

**Lemma 5.5.3.** *Multiplication by  $1 - U_p$  is injective on  $H^1(\mathcal{O}, \mathcal{T}_\theta^\dagger(1))$ .*

*Proof.* As in the proof of Lemma 5.1.2, it suffices to see that  $\mathcal{T}_\theta^\dagger / (U_p - 1)\mathcal{T}_\theta^\dagger(1)$  has trivial  $G_{\mathbb{Q}}$ -invariants. Note that  $\mathcal{T}_{\text{quo}}^\dagger / (U_p - 1)\mathcal{T}_{\text{quo}}^\dagger(1)$  has no  $G_{\mathbb{Q}_p}$ -fixed elements, as the invariant group injects into the trivial kernel of  $1 - U_p$  on  $H^1(\mathbb{Q}_p, \mathcal{T}_{\text{quo}}^\dagger(1)) \cong \mathfrak{S}_\theta^\dagger$

We claim that the quotient  $\mathcal{T}_{\text{sub}}^\dagger / (U_p - 1)\mathcal{T}_{\text{sub}}^\dagger(1)$  has no elements that are fixed by  $G_{\mathbb{Q}}$  inside  $\mathcal{T}_\theta^\dagger / (U_p - 1)\mathcal{T}_\theta^\dagger(1)$ . We know from the proof of Lemma 5.1.2 that for any  $x \in \mathcal{T}_{\text{sub}} - (U_p - 1)\mathcal{T}_{\text{sub}}(1)$ , the group  $x\mathcal{T}_{\text{sub}} / (xm + (U_p - 1)\mathcal{T}_{\text{sub}}(1))$  is isomorphic to a quotient of  $Q/mQ(1)$ , hence can be viewed as a  $G_{\mathbb{Q}}$ -module with action factoring through  $\text{Gal}(\mathbb{Q}(\mu_{Np})/\mathbb{Q})$  via  $\omega^2\theta^{-1}$ . Since  $G_{\mathbb{Q}}$  acts trivially on  $\Lambda^t$  modulo the maximal ideal of  $\Lambda$ , any  $y \in \mathcal{T}_\theta^\dagger - (U_p - 1)\mathcal{T}_\theta^\dagger(1)$  is then similarly acted on by  $G_{\mathbb{Q}}$  through  $\omega^2\theta^{-1}$  inside the quotient  $y\mathcal{T}_{\text{sub}}^\dagger / (y\mathfrak{M} + (U_p - 1)\mathcal{T}_{\text{sub}}^\dagger(1))$ , where  $\mathfrak{M}$  is the unique maximal ideal of  $\Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathfrak{h}_\theta$ . Since  $\theta \neq \omega^2$ , we have the result.  $\square$

A positive answer to Question 5.5.2 appears likely to be too much to hope for in general. However, if it does hold, so does Conjecture 3.1.9.

**Proposition 5.5.4.** *If a map  $z^\dagger$  as in Question 5.5.2 exists, then the reduction  $\bar{z}^\dagger : \Lambda \hat{\otimes}_{\mathbb{Z}_p} P \rightarrow H^1(\mathcal{O}, T^\dagger(1))$  of  $z^\dagger$  modulo  $I$  satisfies the conditions of Theorem 5.5.1, and in particular Conjecture 3.1.9 holds.*

*Proof.* By construction, we have that the composition of  $(1 - U_p)z^\dagger$  with the map

$$H^1(\mathcal{O}, \mathcal{T}_\theta^\dagger(1)) \rightarrow H^1(\mathcal{O}, \mathcal{T}_\theta(1))$$

is  $(1 - U_p)z^\sharp$ . By Lemma 5.1.2, we see that the composition of  $z^\dagger$  with the latter map is  $z^\sharp$ . In particular, the first diagram in Theorem 5.5.1 commutes.

Moreover, since  $(1 - U_p)z_{\text{quo}}^\dagger$  agrees with the composition of  $z_{\text{quo}}$  with the map

$$H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, \mathcal{T}_{\text{quo}}(1)) \rightarrow H^1(\mathbb{Q}_p, \mathcal{T}_{\text{quo}}^\dagger(1)).$$

and  $1 - U_p$  has trivial kernel on  $\mathfrak{S}_\theta^*$ , we have that the composition of  $z^\dagger$  with the map

$$H^1(\mathcal{O}, \mathcal{T}_\theta^\dagger(1)) \rightarrow H^1(\mathbb{Q}_p, \mathcal{T}_{\text{quo}}^\dagger(1))$$

is  $z_{\text{quo}}^\dagger$ . That is, the second diagram in Theorem 5.5.1 commutes.  $\square$

## 6 Test case

We explore the feasibility of the equivalent conditions to Conjecture 3.1.9 found in Theorem 5.5.1, working with cyclotomic elements in place of Beilinson-Kato elements. We find, somewhat reassuringly, that an analogue of the conditions of Theorem 5.5.1 holds in this setting.

On the other hand, an analogue of the stronger Question 5.5.2, which amounts to a norm relation for a good choice of  $z^\dagger$ , has a potential obstruction. We show that this norm relation does hold if an even eigenspace of the completely split Iwasawa module vanishes.

### 6.1 Notation

Let us first introduce changes to our notation from the previous sections. Most importantly, we now allow our prime  $p$  to divide  $\varphi(N)$ . That is, we let  $p$  be an odd prime, and we let  $N \geq 3$  be a positive integer with  $p \nmid N$ . Let  $\Delta = (\mathbb{Z}/Np\mathbb{Z})^\times$  as before, which we identify with  $\text{Gal}(\mathbb{Q}(\mu_{Np^\infty})/\mathbb{Q}_\infty) \cong \text{Gal}(\mathbb{Q}(\mu_{Np})/\mathbb{Q})$ .

#### Definition 6.1.1.

- a. Let  $\Delta_p$  and  $\Delta'$  be the Sylow  $p$ -subgroup of  $\Delta$  and its prime-to- $p$  order complement, respectively.
- b. Let  $\theta: \Delta' \rightarrow \overline{\mathbb{Q}}_p^\times$  be a nontrivial even character of  $\Delta'$  which is trivial on decomposition at  $p$  and primitive at all primes dividing  $N$ .
- c. Let  $R_\theta$  be the  $\mathbb{Z}_p[\Delta_p]$ -algebra of values of  $\theta$ , which we then view as a  $\mathbb{Z}_p[\Delta]$ -module with  $\Delta'$  acting through  $\theta$ .
- d. Let  $\Lambda_\theta = R_\theta[[\Gamma]] \cong R_\theta[[X]]$ , where  $\Gamma \cong \text{Gal}(\mathbb{Q}(\mu_{Np^\infty})/\mathbb{Q}(\mu_{Np}))$ , and  $X$  is as before.

- e. Let  $\mathcal{R} = R_\theta^i$  be the  $\mathbb{Z}_p[\Delta]$ -module that is  $R_\theta$  endowed with the inverse of the Galois action described above.
- f. The  $\theta$ -part  $M_\theta$  of a  $\mathbb{Z}_p[\Delta]$ -module  $M$  is the  $R_\theta$ -module  $M_\theta = M \otimes_{\mathbb{Z}_p[\Delta]} R_\theta$ .

*Remark 6.1.2.* Our choice of  $\mathcal{R}$  is made so that

$$H^i(\mathcal{O}, \mathcal{R}(1)) \cong H^i(\mathbb{Z}[\frac{1}{Np}], \mu_{Np}, \mathbb{Z}_p(1))_\theta \cong H^i(\mathbb{Z}[\frac{1}{Np}], \mu_N, \mathbb{Z}_p(1))_\theta$$

for any  $i \geq 0$  by Shapiro's lemma, and similarly for Iwasawa cohomology.

We shall also use the following.

**Definition 6.1.3.**

- a. Let  $\sigma$  denote the image of the Frobenius  $\text{Fr}_p$  at  $p$  in  $\Delta_p$ .
- b. Let  $R_{\sigma=1}$  denote the maximal quotient and  $R^{\sigma=1}$  the maximal submodule of  $R_\theta$  on which  $\sigma$  acts trivially.
- c. Let  $Y$  (resp.,  $X$ ) denote the Galois group of the maximal completely locally split (resp., unramified) abelian pro- $p$  extension of  $\mathbb{Q}(\mu_{Np^\infty})$ .
- d. Let  $\mathfrak{X}$  denote the Galois group of the maximal abelian, unramified outside  $Np$ , pro- $p$  extension of  $\mathbb{Q}(\mu_{Np^\infty})$ .
- e. Let  $\mathcal{E}$  (resp.,  $\mathcal{C}$ ) denote the group of norm compatible systems of  $p$ -completions of global units (resp., cyclotomic units) in the tower  $\mathbb{Q}(\mu_{Np^\infty})/\mathbb{Q}$ .

## 6.2 Zeta and Coleman maps

We now take our zeta map as having image the  $\theta$ -eigenspace of the cyclotomic units.

**Definition 6.2.1.**

- a. The *zeta map*  $z$  is the  $\Lambda_\theta$ -module homomorphism

$$z: \Lambda_\theta \rightarrow H_{\text{Iw}}^1(\mathcal{O}_\infty, \mathcal{R}(1))$$

that sends 1 to the projection of the norm compatible sequence  $(1 - \zeta_{Np^r})_{r \geq 1}$ .

- b. We define a  $\mathbb{Z}_p[\Delta]$ -module homomorphism

$$z^\sharp: R_\theta \rightarrow H^1(\mathcal{O}, \mathcal{R}(1))$$

as the unique such map taking 1 to the projection of  $1 - \zeta_N$ .

We use  $z_{\text{quo}}$  and  $z_{\text{quo}}^\sharp$  to denote the restrictions of  $z$  and  $z^\sharp$  to the cohomology of  $G_{\mathbb{Q}_p}$ .

*Remark 6.2.2.* The zeta map and its ground level analogue satisfy the well-known norm relation

$$\begin{array}{ccc} \Lambda_\theta & \xrightarrow{z} & H_{\text{Iw}}^1(\mathcal{O}_\infty, \mathcal{R}(1)) \\ \downarrow z^\sharp \circ \text{ev}_0 & & \downarrow \\ H^1(\mathcal{O}, \mathcal{R}(1)) & \xrightarrow{1-\sigma^{-1}} & H^1(\mathcal{O}, \mathcal{R}(1)). \end{array}$$

among cyclotomic units.

**Definition 6.2.3.** We let  $\xi \in \Lambda_\theta$  be the unique element satisfying

$$\tilde{\rho}(\xi(u^{1-s} - 1)) = L_p(\theta\rho, s)$$

for all  $s \in \mathbb{Z}_p$  and  $p$ -adic characters  $\rho$  of  $\Delta_p$ , where we use  $\tilde{\rho}$  to denote the map  $R_\theta \rightarrow \overline{\mathbb{Q}_p}$  induced by  $\rho$ .

We note the following equivariant formulation of Iwasawa's theorem.

*Remark 6.2.4.* As an  $R_\theta$ -module,  $D(\mathcal{R})$  is free of rank 1, and it can be identified with  $R_\theta$  as a  $\mathbb{Z}_p$ -algebra after a choice of normal basis of the valuation ring of the unramified extension of  $\mathbb{Q}(\mu_p)$  defined by the decomposition group of  $\Delta_p$ . We can and do choose this identification such that the Coleman map

$$\text{Col} = \text{Col}_{\mathcal{R}}: H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, \mathcal{R}(1)) \rightarrow X^{-1}\Lambda_\theta$$

satisfies  $\text{Col} \circ z = \xi$ .

To shorten notation, let us write  $\mathfrak{C}$  for the image  $\mathfrak{C}(\mathcal{R})$  of  $\text{Col}$  and similarly with superscripts adorning  $\mathfrak{C}$ . Consider the Coleman map

$$\text{Col}^\dagger: H^1(\mathbb{Q}_p, \mathcal{R}^\dagger(1)) \rightarrow \mathfrak{C}^*$$

for  $A = \mathcal{R}$  and  $\alpha = \xi$  of Theorem 4.2.4. The analogous argument to that of Proposition 4.3.4 yields the following.

**Proposition 6.2.5.** *There exists a canonical  $\Lambda_\theta$ -module homomorphism*

$$z_{\text{quo}}^\dagger: R_\theta \rightarrow H^1(\mathbb{Q}_p, \mathcal{R}^\dagger(1))$$

such that  $\overline{\text{Col}}^\dagger \circ z_{\text{quo}}^\dagger = 1$  and such that the diagram

$$\begin{array}{ccc} \Lambda_\theta & \xrightarrow{z_{\text{quo}}} & H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, \mathcal{R}(1)) \\ \downarrow z_{\text{quo}}^\dagger \circ \text{ev}_0 & & \downarrow \\ H^1(\mathbb{Q}_p, \mathcal{R}^\dagger(1)) & \xrightarrow{1-\sigma^{-1}} & H^1(\mathbb{Q}_p, \mathcal{R}^\dagger(1)) \end{array}$$

commutes. Moreover,  $z_{\text{quo}}^{\sharp}$  is the composition of  $z_{\text{quo}}^{\dagger}$  with the map

$$H^1(\mathbb{Q}_p, \mathcal{R}^{\dagger}(1)) \rightarrow H^1(\mathbb{Q}_p, \mathcal{R}(1)).$$

*Proof.* Identifying  $H^1(\mathbb{Q}_p, \mathcal{R}^{\dagger}(1))$  with  $\mathfrak{C}^*$  via  $\text{Col}^{\dagger}$ , we define  $z_{\text{quo}}^{\dagger}$  to be as the pushout map  $R_{\theta} \cong D(\mathcal{R}) \rightarrow \mathfrak{C}^*$ . By definition of  $\mathfrak{C}^*$ , following this by  $1 - \sigma^{-1}$ , we get the composition

$$R_{\theta} \xrightarrow{\xi} \mathfrak{C}^{\dagger} \rightarrow \mathfrak{C}^*,$$

which is to say, recalling Remark 6.2.4, the composition of  $z_{\text{quo}}$  with  $H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, \mathcal{R}(1)) \rightarrow H^1(\mathbb{Q}_p, \mathcal{R}^{\dagger}(1))$ .  $\square$

### 6.3 Brief cohomological study

We describe the structure of some relevant cohomology groups.

**Lemma 6.3.1.** *For each prime  $\ell \nmid N$ , the cohomology groups  $H^i(\mathbb{Q}_{\ell}, \mathcal{R}(1))$ ,  $H^i(\mathbb{Q}_{\ell}, \mathcal{R}^{\dagger}(1))$ , and  $H_{\text{Iw}}^i(\mathbb{Q}_{\ell}, \Lambda_{\theta}^1(1))$  for  $i \in \{1, 2\}$  are all trivial.*

*Proof.* By Shapiro's lemma, the group  $H^i(\mathbb{Q}_{\ell}, \mathcal{R}(1))$  is isomorphic to the  $\theta$ -eigenspace of the product of the groups  $H^i(\mathbb{Q}_{\ell}(\mu_{Np}), A_{\theta}(1))$  over primes over  $\ell$  in the field  $\mathbb{Q}(\mu_{Np})$ , where  $A_{\theta}$  is the  $\mathbb{Z}_p$ -algebra of  $\theta$ -values with the trivial action of Galois. Since the pro- $p$  completion of  $\mathbb{Q}_{\ell}(\mu_{Np})^{\times}$  is generated by a uniformizer, each of these first cohomology groups is isomorphic to  $A_{\theta}$  via the Kummer isomorphism. The second cohomology groups are also isomorphic to  $A_{\theta}$  via the invariant map. Since inertia at  $\ell$  in  $\text{Gal}(\mathbb{Q}_{\ell}(\mu_{Np})/\mathbb{Q}_{\ell})$  acts trivially on this product and  $\theta$  is primitive at  $\ell$ , the  $\theta$ -eigenspace of the product is zero.

We have an exact sequence

$$0 \rightarrow H^1(\mathbb{Q}_{\ell}, \Lambda_{\theta}^1(1))/X\xi \rightarrow H^1(\mathbb{Q}_{\ell}, \mathcal{R}^{\dagger}(1)) \rightarrow H^2(\mathbb{Q}_{\ell}, \Lambda_{\theta}(1))[X\xi] \rightarrow 0,$$

and  $H^2(\mathbb{Q}_{\ell}, \mathcal{R}^{\dagger}(1))$  is a quotient of  $H^2(\mathbb{Q}_{\ell}, \Lambda_{\theta}(1))$ . The groups  $H^i(\mathbb{Q}_{\ell}, \Lambda_{\theta}^1(1))$  are zero for  $i \in \{1, 2\}$ , each being isomorphic to the  $\theta$ -eigenspace of the product of the groups

$$H_{\text{Iw}}^i(\mathbb{Q}_{\ell}(\mu_{Np^{\infty}}), A_{\theta}(1)) \cong A_{\theta}$$

over primes of  $\mathbb{Q}(\mu_{Np^{\infty}})$  over  $\ell$ .  $\square$

The invariant map provides an isomorphism

$$\text{inv}: H_{\text{Iw}}^2(\mathbb{Q}_{p,\infty}, \mathcal{R}(1)) \xrightarrow{\sim} R_{\sigma=1},$$

since  $R_{\sigma=1}$  is the maximal unramified quotient of  $R_{\theta}$ . Note also that  $H_{c,\text{Iw}}^3(\mathcal{O}_{\infty}, \mathcal{R}(1)) = 0$  since  $\Delta'$  acts on  $\mathcal{R}$  through the nontrivial prime-to- $p$  order character  $\theta^{-1}$ .

**Lemma 6.3.2.** *We have an exact sequence of  $\Lambda$ -modules*

$$0 \rightarrow Y_\theta \rightarrow H_{\text{Iw}}^2(\mathcal{O}_\infty, \mathcal{R}(1)) \rightarrow R_{\sigma=1} \rightarrow 0.$$

*Proof.* This is immediate from the Poitou-Tate sequence, the invariant map for  $\mathbb{Q}_{p,\infty}$ , the triviality of  $H^2(\mathbb{Q}_\ell, \Lambda_\theta^1(1))$  for  $\ell \mid N$  of Lemma 6.3.1, and the triviality of  $H_{c,\text{Iw}}^3(\mathcal{O}_\infty, \mathcal{R}(1))$ .  $\square$

**Lemma 6.3.3.** *We have  $H_c^1(\mathcal{O}, \mathcal{R}(1)) = 0$  and  $H_c^i(\mathcal{O}, \mathcal{R}^\dagger(1)) \cong \mathfrak{X}_\theta$  for  $i \in \{1, 2\}$ .*

*Proof.* We employ some well-known results of classical Iwasawa theory. Recall that  $\mathfrak{X}_\theta$  is pseudo-isomorphic to  $(Y^1(1))_\theta$ , where  $Y^1$  is  $Y$  with the inverse  $\text{Gal}(\mathbb{Q}(\mu_{Np^\infty})/\mathbb{Q})$ -action. (In this eigenspace, there is no difference between  $Y$  and the unramified Iwasawa module.) The group  $Y^1(1)$  is annihilated by  $\xi$  by Stickelberger theory. Since  $\mathfrak{X}_\theta$  has no finite  $\Lambda_\theta$ -submodules, it too is annihilated by  $\xi$ .

Since  $H_{c,\text{Iw}}^3(\mathcal{O}_\infty, \mathcal{R}(1))$  is trivial and

$$H_{c,\text{Iw}}^2(\mathcal{O}_\infty, \mathcal{R}(1)) \cong (H^1(\mathbb{Z}_\infty[\mu_{Np}, \frac{1}{Np}], \mathbb{Q}_p/\mathbb{Z}_p)^\vee)_\theta \cong \mathfrak{X}_\theta$$

by Poitou-Tate duality, we have

$$H_c^2(\mathcal{O}, \mathcal{R}^\dagger(1)) \cong H_{c,\text{Iw}}^2(\mathcal{O}_\infty, \mathcal{R}(1))/X\xi \cong \mathfrak{X}_\theta/X\xi \cong \mathfrak{X}_\theta.$$

Next, note that we have an exact sequence

$$0 \rightarrow H_{c,\text{Iw}}^1(\mathcal{O}_\infty, \mathcal{R}(1))/X\xi \rightarrow H_c^1(\mathcal{O}, \mathcal{R}^\dagger(1)) \rightarrow H_{c,\text{Iw}}^2(\mathcal{O}_\infty, \mathcal{R}(1))[X\xi] \rightarrow 0,$$

and  $H_{c,\text{Iw}}^1(\mathcal{O}_\infty, \mathcal{R}(1))$  vanishes by the weak Leopoldt conjecture. Thus,  $H_c^1(\mathcal{O}, \mathcal{R}^\dagger(1)) \cong \mathfrak{X}_\theta$  by the above description of the rightmost group in the sequence. Similarly,  $H_c^1(\mathcal{O}, \mathcal{R}(1))$  is trivial by the Leopoldt conjecture for abelian fields.  $\square$

## 6.4 Questions and answers

We first show that an analogue of Proposition 5.5.2 does indeed hold.

**Proposition 6.4.1.** *There exists a  $\Lambda_\theta$ -module homomorphism*

$$z^\dagger: \Lambda_\theta \rightarrow H^1(\mathcal{O}, \mathcal{R}^\dagger(1))$$

*such that the diagrams*

$$\begin{array}{ccc} \Lambda_\theta & \xrightarrow{z^\dagger} & H^1(\mathcal{O}, \mathcal{R}^\dagger(1)) \\ \downarrow & & \downarrow \\ R_\theta & \xrightarrow{z^\ddagger} & H^1(\mathcal{O}, \mathcal{R}(1)) \end{array} \quad \text{and} \quad \begin{array}{ccc} \Lambda_\theta & \xrightarrow{z^\dagger} & H^1(\mathcal{O}, \mathcal{R}^\dagger(1)) \\ \downarrow & & \downarrow \\ R_\theta & \xrightarrow{z_{\text{quo}}^\dagger} & H^1(\mathbb{Q}_p, \mathcal{R}^\dagger(1)) \end{array}$$

*commute.*

*Proof.* Poitou-Tate and Lemma 6.3.1 provide a map of exact sequences

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_c^1(\mathcal{O}, \mathcal{R}^\dagger(1)) & \longrightarrow & H^1(\mathcal{O}, \mathcal{R}^\dagger(1)) & \longrightarrow & H^1(\mathbb{Q}_p, \mathcal{R}^\dagger(1)) & \longrightarrow & H_c^2(\mathcal{O}, \mathcal{R}^\dagger(1)) \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H_c^1(\mathcal{O}, \mathcal{R}(1)) & \longrightarrow & H^1(\mathcal{O}, \mathcal{R}(1)) & \longrightarrow & H^1(\mathbb{Q}_p, \mathcal{R}(1)) & \longrightarrow & H_c^2(\mathcal{O}, \mathcal{R}(1)).
\end{array}$$

As already noted,  $z^\sharp$  induces a map  $z_{\text{quo}}^\sharp$  that lifts to a map  $z_{\text{quo}}^\dagger$ . The image of  $z_{\text{quo}}^\dagger$  lies by definition in the  $\Gamma$ -invariant group of  $H^1(\mathbb{Q}_p, \mathcal{R}^\dagger(1))$  and therefore maps to  $H_c^2(\mathcal{O}, \mathcal{R}^\dagger(1))^\Gamma = \mathfrak{X}_\theta^\Gamma$  by Lemma 6.3.3. But  $\mathfrak{X}_\theta^\Gamma = 0$  by weak Leopoldt (cf. [NSW, (11.3.3) and (11.3.5)]).

Thus, there exists an element  $x \in H^1(\mathcal{O}, \mathcal{R}^\dagger(1))$  with image  $z_{\text{quo}}^\dagger(1) \in H^1(\mathbb{Q}_p, \mathcal{R}^\dagger(1))$ , which since  $H_c^1(\mathcal{O}, \mathcal{R}(1)) = 0$  by Lemma 6.3.3, necessarily then also has image  $z^\sharp(1) \in H^1(\mathcal{O}, \mathcal{R}(1))$ . We can then take  $z^\dagger$  as the unique  $\Lambda_\theta$ -module homomorphism with  $z^\dagger(1) = x$ .  $\square$

Note that  $z^\dagger$  in Proposition 6.4.1 is unique only up to an element of  $H_c^2(\mathcal{O}, \mathcal{R}^\dagger(1)) \cong \mathfrak{X}_\theta$  (by Lemma 6.3.3). The analogue of Question 5.5.2 is the following.

*Question 6.4.2.* Does there exist a  $\Lambda_\theta$ -module homomorphism

$$z^\dagger: \Lambda_\theta \rightarrow H^1(\mathcal{O}, \mathcal{R}^\dagger(1))$$

as in Proposition 6.4.1 such that the diagram

$$\begin{array}{ccc}
\Lambda_\theta & \xrightarrow{z} & H_{\text{Iw}}^1(\mathcal{O}_\infty, \mathcal{R}(1)) \\
\downarrow z^\dagger & & \downarrow \\
H^1(\mathcal{O}, \mathcal{R}^\dagger(1)) & \xrightarrow{1-\sigma^{-1}} & H^1(\mathcal{O}, \mathcal{R}^\dagger(1))
\end{array}$$

commutes?

*Remark 6.4.3.* It is easy enough to construct a map  $y^\dagger: R_\theta \rightarrow H_{\text{Iw}}^1(\mathcal{O}_\infty, \mathcal{R}(1))$  such that  $(1 - \sigma^{-1})y^\dagger \circ \text{ev}_0$  is the image of  $\xi(0)z$ . For this, compose  $z^\sharp$  with the multiplication-by- $\xi$  map  $H^1(\mathcal{O}, \mathcal{R}(1)) \rightarrow H^1(\mathcal{O}_\infty, \mathcal{R}(1))$ . This, however, is not ideal: it is the analogue of multiplying by the derivative  $\xi'$  in the setting of the other sections of this paper. In general, one cannot do better than this if we ask for a map from  $R_\theta$ , rather than  $\Lambda_\theta$ .

If we suppose that  $Y_\theta = 0$ , then a map  $z^\dagger$  as in Question 6.4.2 does indeed exist. To see this, we first compute the relevant Iwasawa modules under this assumption (assuming some standard results of classical Iwasawa theory without reference).



**Lemma 6.4.4.** *If  $Y_\theta = 0$ , then the Coleman map fits in an isomorphism of exact sequences*

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_{\text{Iw}}^1(\mathcal{O}_\infty, \mathcal{R}(1)) & \longrightarrow & H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, \mathcal{R}(1)) & \longrightarrow & \mathfrak{X}_\theta \longrightarrow 0 \\
& & \downarrow \wr & & \text{Col} \downarrow \wr & & \downarrow \wr \\
0 & \longrightarrow & \mathfrak{C} & \xrightarrow{\xi} & \mathfrak{C} & \longrightarrow & \mathfrak{C}/\xi\mathfrak{C} \longrightarrow 0
\end{array}$$

of  $\Lambda_\theta$ -modules. Moreover, we have a  $\Lambda_\theta$ -module isomorphism  $X_\theta \cong R^{\sigma=1}/\xi(0)R^{\sigma=1}$ .

*Proof.* The exactness of the top row in the diagram follows from the Poitou-Tate sequence, the three lemmas of Section 6.3, and, for right exactness, our assumption that  $Y_\theta = 0$ . Note that Col gives an isomorphism  $H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, \mathcal{R}(1)) \cong \mathfrak{C}$ . Since  $Y_\theta = 0$ , the Iwasawa module  $\mathfrak{X}_\theta$  is then a quotient of  $\mathfrak{C}$ . By Stickelberger theory, we know it is annihilated by  $\xi$ . The main conjecture tells us that the characteristic ideal of the maximal quotient of  $\mathfrak{X}_\theta$  upon which  $\Delta_p$  acts through a given character is generated by the image of  $\xi$  in the resulting Iwasawa algebra. Since  $\mathfrak{C}/\xi\mathfrak{C}$  has this property, and  $\mathfrak{X}_\theta$  has no finite  $\Lambda_\theta$ -submodules, we must have  $\mathfrak{X}_\theta \cong \mathfrak{C}/\xi\mathfrak{C}$ .

Since  $Y_\theta = 0$ , the group  $X_\theta$  is finite. Then  $\mathcal{E}_\theta = \mathcal{C}_\theta$  by a standard argument, and  $\mathcal{C}_\theta \cong \Lambda_\theta$ , generated by the projection of  $(1 - \zeta_{Np^r})_r$ . As  $\text{Col} \circ z = \xi$ , the resulting composition

$$\Lambda_\theta \xrightarrow{\sim} \mathcal{E}_\theta \rightarrow H_{\text{Iw}}^1(\mathbb{Q}_{p,\infty}, \mathcal{R}(1)) \xrightarrow{\sim} \mathfrak{C}$$

is given by multiplication by  $\xi$ . Kummer theory then gives a map of exact sequences

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Lambda_\theta & \longrightarrow & H_{\text{Iw}}^1(\mathcal{O}_\infty, \mathcal{R}(1)) & \longrightarrow & R^{\sigma=1} \longrightarrow X_\theta \longrightarrow 0, \\
& & \downarrow \xi & & \downarrow & & \parallel \\
0 & \longrightarrow & \Lambda_\theta & \longrightarrow & \mathfrak{C} & \longrightarrow & R^{\sigma=1} \longrightarrow 0,
\end{array}$$

where the upper map to  $R^{\sigma=1}$  is given by the valuations at the primes over  $p$  (cf. [NSW, (11.3.10)]). Since  $\mathfrak{C}$  has no  $\Lambda$ -torsion, any element  $a \in H_{\text{Iw}}^1(\mathcal{O}_\infty, \mathcal{R}(1))$  with  $Xa = b \in \Lambda_\theta$ , is taken to  $X^{-1}\xi b \in \mathfrak{C}$ , and therefore has image  $\xi(0)b \in R^{\sigma=1}$ . In other words, we have a surjection  $X_\theta \rightarrow R^{\sigma=1}/\xi(0)R^{\sigma=1}$ . At the same time,  $\xi$  annihilates  $\mathfrak{X}_\theta$ , so  $\xi(0)$  annihilates  $X_\theta$ , and this map is an isomorphism.

We then have a surjective map  $H_{\text{Iw}}^1(\mathcal{O}_\infty, \mathcal{R}(1)) \rightarrow R^{\sigma=1}$  given by  $\xi(0)^{-1}$  times the valuation maps, and the resulting map  $R^{\sigma=1} \rightarrow R^{\sigma=1}$  becomes multiplication by  $\xi(0)$ , or  $\xi$ . This identifies  $H_{\text{Iw}}^1(\mathcal{O}_\infty, \mathcal{R}(1))$  with  $\xi\mathfrak{C}$  as a subgroup of  $\mathfrak{C}$ . In other words, we have an isomorphism  $H_{\text{Iw}}^1(\mathcal{O}_\infty, \mathcal{R}(1)) \cong \mathfrak{C}$  such that that resulting map  $\mathfrak{C} \rightarrow \mathfrak{C}$  is multiplication by  $\xi$ .  $\square$

Note that the composition of  $z$  with the isomorphism  $H_{\text{Iw}}^1(\mathcal{O}_\infty, \mathcal{R}(1)) \rightarrow \mathfrak{C}$  of Lemma 6.4.4 is the canonical injection  $\Lambda_\theta \rightarrow \mathfrak{C}$ .

**Proposition 6.4.5.** *Suppose that  $Y_\theta = 0$ . Then Question 6.4.2 has a positive answer.*

*Proof.* Since  $\mathfrak{C}$  has no  $\Lambda$ -torsion and  $\mathfrak{X}_\theta$  is annihilated by  $X\xi$ , the short exact sequence of Lemma 6.4.4 gives rise to the first row in the commutative diagram

$$\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{X}_\theta & \longrightarrow & \mathfrak{C}^\dagger & \xrightarrow{\xi} & \mathfrak{C}^\dagger & \longrightarrow & \mathfrak{X}_\theta \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & \mathfrak{X}_\theta & \longrightarrow & H^1(\mathcal{O}, \mathcal{R}^\dagger(1)) & \longrightarrow & \mathfrak{C}^\star & \longrightarrow & \mathfrak{X}_\theta \longrightarrow 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & H_{\text{Iw}}^2(\mathcal{O}_\infty, \mathcal{R}(1)) & \xrightarrow{\sim} & R_{\sigma=1} & & \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & & 
\end{array}$$

with exact rows and columns. The exactness of the middle row is by the Poitou-Tate sequence, Theorem 4.2.4, and Lemmas 6.3.1 and 6.3.3, while the isomorphism of the final row is Lemma 6.3.2.

Let  $x \in H^1(\mathcal{O}, \mathcal{R}^\dagger(1))$  map to  $1 \in R_{\sigma=1}$ , and set  $y = (1 - \sigma^{-1})x \in \mathfrak{C}^\dagger$ , using the identification given by Lemma 6.4.4. By definition of  $\mathfrak{C}^\star$ , the image  $x_{\text{loc}}$  of  $x$  in  $\mathfrak{C}^\star$  has the property that  $y_{\text{loc}} = (1 - \sigma^{-1})x_{\text{loc}} = \xi \in \mathfrak{C}^\dagger$ . This forces  $\xi y = \xi$  and hence  $y \equiv 1 \pmod{X}$  in the image of  $\Lambda_\theta$  in  $\mathfrak{C}^\dagger$ . Choose  $\lambda \in \Lambda_\theta$  such that  $\lambda y = 1$ . We then define  $z^\dagger$  as the unique  $\Lambda_\theta$ -module homomorphism with  $z^\dagger(1) = \lambda x$ .

By construction,  $x_{\text{loc}} = \text{Col}^\dagger \circ z_{\text{quo}}^\dagger(1)$ . Since  $z_{\text{quo}}^\dagger(1)$  is  $\Gamma$ -fixed, we have that  $\lambda x_{\text{loc}} = x_{\text{loc}}$ , and  $z^\dagger$  restricts to  $z_{\text{quo}}^\dagger$ . Moreover,  $(1 - \sigma^{-1})z^\dagger(1) = 1 \in \mathfrak{C}^\dagger$  is the image of  $z(1)$ , so we have the commutativity of the diagram in Question 6.4.2. Finally,  $(1 - \sigma^{-1})z^\dagger(1)$  has image  $(1 - \sigma^{-1})z^\sharp(1) \in H^1(\mathcal{O}, \mathcal{R}(1))$ , and the norm of  $z^\sharp(1)$  under the subgroup generated by  $\sigma$  is trivial, so  $z^\dagger(1)$  maps to  $z^\sharp(1)$  as well.  $\square$

Without assuming that  $Y_\theta = 0$ , the existence of  $z^\dagger$  as in Question 6.4.2 requires the splitting of the exact sequence of Lemma 6.3.2 as  $R_\theta$ -modules.

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