Iwasawa modules of higher codimension

Romyar Sharifi
joint with F. Bleher, T. Chinburg, R. Greenberg, M. Kakde, M. Taylor

University of California, Los Angeles

March 22, 2019

Notation

- E CM (or totally real) number field
- E^+ maximal totally real subfield, $d = [E^+ : \mathbb{Q}]$
- p odd prime such that all primes over it split in E/E^+
- F/E finite abelian with $\mu_p \subset F$ and $p \nmid [F : E]$ and $\Delta = \mathrm{Gal}(F/E)$
- $\tilde{E}$ compositum of all $\mathbb{Z}_p$-extensions of E
- $\Gamma = \mathrm{Gal}(\tilde{E}/E) \cong \mathbb{Z}_p^r$, where $r \geq r_2(E) + 1$
- $K = F\tilde{E}$ and $H = \mathrm{Gal}(K/E) \cong \Gamma \times \Delta$

$$\begin{tikzcd}
 & K \arrow[dl, dash, "\Gamma"'] \arrow[d, dash, "\Delta"] \arrow[ddl, dash, "H"] \\
F \arrow[d, dash, "\Delta"'] & \tilde{E} \arrow[dl, dash, "\Gamma"] \\
E \arrow[d, dash] & \\
E^+ &
\end{tikzcd}$$

Notation (Σ-ramified Iwasawa module X_Σ)

- S_p the set of primes of E over p and $\Omega = \mathbb{Z}_p[\![H]\!]$
- For $\Sigma \subseteq S_p$, let X_Σ be the Ω-module that is the Galois group of maximal abelian pro-p unramified outside Σ extension of K.

Examples

- p-ramified Iwasawa module $\mathfrak{X} = X_{S_p}$
- unramified Iwasawa module $X = X_\varnothing$

Notation (ψ-eigenspace M^ψ)

- $\Lambda = W[\![\Gamma]\!]$ for Witt vectors W of $\overline{\mathbb{F}_p}$
- $\psi\colon \Delta \to W^\times$ abelian character
- For an Ω-module M, the ψ-eigenspace of M is the Λ-module

$$M^\psi = M \otimes_{\mathbb{Z}_p[\Delta]} W$$

via the $\mathbb{Z}_p$-linear map $\mathbb{Z}_p[\Delta] \to W$ induced by ψ.

Hypotheses

- $\omega\colon \Delta \to \mathbb{Z}_p^\times$ unique lift of the mod p cyclotomic character
- For simplicity in the descriptions of our results, we suppose that ψ and $\omega\psi^{-1}$ are nontrivial on all decomposition groups $\Delta_\mathfrak{p}$ for $\mathfrak{p} \in S_p$.
- If E is totally real, let us assume that ψ is totally odd.

Definition

A (p-adic) CM type $\mathcal{S} \subset S_p$ is a set of one among each complex conjugate pair of distinct primes over p. In particular, $\mathcal{S} = \varnothing$ if E totally real.

Remark

For any set Σ of primes containing a CM type $\mathcal{S}$, we have

$$\operatorname{rank}_\Lambda X_\Sigma^\psi = \sum_{\mathfrak{p} \in \Sigma - \mathcal{S}} [E_\mathfrak{p} : \mathbb{Q}].$$

In particular, $\mathfrak{X}^\psi$ has Λ-rank $d = [E^+ : \mathbb{Q}]$, and $X_\mathcal{S}^\psi$ is a torsion Λ-module.

Remark (p-adic L-function)

Attached to a CM type $\mathcal{S}$ and ψ, we have an element $\mathcal{L}_{\mathcal{S},\psi} \in \Lambda$ corresponding to the Katz or Deligne-Ribet p-adic L-function $L_p(\omega\psi^{-1}, s)$

Conjecture (Iwasawa main conjecture)

For a CM type $\mathcal{S}$, the Λ-characteristic ideal of $X_{\mathcal{S}}^{\psi}$ is generated by $\mathcal{L}_{\mathcal{S},\psi}$.

Remark

The main conjecture in this form is known for E totally real (Mazur-Wiles, Wiles) and for E imaginary quadratic (Rubin). Progress in the general CM case is due for instance to Hsieh; this main conjecture was introduced by Hida and Tilouine.

Remark (Redifining $\mathcal{L}_{\mathcal{S},\psi}$)

If the relevant main conjecture has not been shown to hold, we can and do for our purposes instead take $\mathcal{L}_{\mathcal{S},\psi}$ simply to be any choice of generator of the characteristic ideal of $X_{\mathcal{S}}^{\psi}$.

Remark

For $E = \mathbb{Q}$ (or totally real), Leopoldt formulated a reflection principle relating the p-ranks of the ψ and $\omega\psi^{-1}$-eigenspaces of the class group of F.

Passing up the tower, we have what we might call a "reflection sequence."

Proposition

Let $E = \mathbb{Q}$, so ψ is odd and $\Gamma \cong \mathbb{Z}_p$. There is a canonical exact sequence of Λ-modules

$$0 \to \alpha(X^{\omega\psi^{-1}})(1) \to X^{\psi} \to \Lambda/(f) \to (X_{\mathrm{fin}}^{\omega\psi^{-1}})^{\vee}(1) \to 0$$

for some $f \in \Lambda$ dividing $\mathcal{L}_{\varnothing,\psi}$, α denotes the Iwasawa adjoint and $\vee$ the Pontryagin dual, where X_{fin} is the maximal finite submodule of X.

That is, a dual of $X^{\omega\psi^{-1}}$ differs from X^{ψ} by a pseudo-cyclic Λ-module, where "pseudo" here means up to modules having finite length over W.

Remark

Greenberg conjectured that X^{+} is finite, and if this holds, then the sequence in the proposition tells us that X^{ψ} is pseudo-cyclic.

Suppose now that E is imaginary quadratic.
By assumption, $p\mathcal{O}_E = \mathfrak{p}\bar{\mathfrak{p}}$, and we have $\Gamma \cong \mathbb{Z}_p^2$.

Remark

Greenberg conjectures that X^{ψ} is pseudo-null, i.e., of codimension at least 2.

Notation

We replace $\{\mathfrak{p}\}$ by $\mathfrak{p}$ in subscripts, and similarly for $\bar{\mathfrak{p}}$, e.g., $X_{\mathfrak{p}} = X_{\{\mathfrak{p}\}}$.

With G. Pappas, we proved the following:

Theorem (BCGKPST)

Suppose that $X^{\omega\psi^{-1}}$ is pseudo-null. Then there is a canonical exact sequence of Λ-modules

$$0 \to X^{\psi} \to \Lambda/(\mathcal{L}_{\mathfrak{p},\psi}, \mathcal{L}_{\bar{\mathfrak{p}},\psi}) \to \alpha(X^{\omega\psi^{-1}})(1) \to 0,$$

where α denotes an "Iwasawa adjoint" for pseudo-null Λ-modules.

This is again a "reflection sequence", where now X^{ψ} and $X^{\omega\psi^{-1}}$ are both conjecturally pseudo-null, as opposed to just one for $E = \mathbb{Q}$.

A corollary

Definition

Let M be a finitely generated Λ-module of codimension n. Let X_n denote the set of height n primes of Λ. The nth Chern class $c_n(M)$ of M is

$$c_n(M) = \sum_{\mathfrak{q} \in X_n} \ell_{\mathfrak{q}}(M)[\mathfrak{q}]$$

in the free abelian group on X_n, where $\ell_{\mathfrak{q}}$ is length over $\Lambda_{\mathfrak{q}}$.

Remark

$c_1(M)$ is the divisor corresponding to the characteristic ideal of M

Definition

Let $\mathcal{O}$ be a compact $\mathbb{Z}_p$-algebra. For an $\mathcal{O}[\![H]\!]$-module M, let M^ι denote the $\mathcal{O}$-module M with the continuous $\mathcal{O}$-linear action of H given by $(h, m) \mapsto h^{-1}m$.

Corollary

Suppose E is imaginary quadratic and X^ψ and $X^{\omega\psi^{-1}}$ are pseudo-null. Then

$$c_2(\Lambda/(\mathcal{L}_{\mathfrak{p},\psi}, \mathcal{L}_{\bar{\mathfrak{p}},\psi})) = c_2(X^\psi) + c_2((X^{\omega\psi^{-1}})^\iota(1)).$$

An exact sequence of Ext-groups

Notation (Ext-groups and Iwasawa cohomology)

- For an Ω-module M, we set
$$E^i(M) = \mathrm{Ext}^i_\Omega(M^\iota, \Omega).$$
For a Λ-module M, we also set $E^i(M) = \mathrm{Ext}^i_\Lambda(M^\iota, \Lambda)$. Let $M^* = E^0(M)$.
- $H^i_{\mathrm{Iw}}(K, T)$ is the ith Iwasawa cohomology group for the maximal p-ramified extension of K with T-coefficients. Set $H^i_{\mathrm{Iw}} = H^i_{\mathrm{Iw}}(K, \mathbb{Z}_p(1))$.
- X' maximal quotient of X in which all primes split completely

Remarks

- For a finitely generated Ω-module M, the codimension of $E^n(M)$ is at least n, and if M has codimension n, then $E^i(M) = 0$ for all $i < n$.
- The Iwasawa adjoint α in the theorem is just E^2.

Proposition (BCGKPST)

There is a canonical exact sequence of Ω-modules
$$0 \to E^1(H^2_{\mathrm{Iw}})(1) \to \mathfrak{X} \to \mathfrak{X}^{**} \to E^2(H^2_{\mathrm{Iw}})(1) \to \mathbb{Z}_p.$$

H^2_{Iw} and inertia groups

Remark

Let $H_{\mathfrak{p}}$ denote the decomposition group at $\mathfrak{p}$ in H. We have an exact sequence

$$0 \to X' \to H^2_{\mathrm{Iw}} \to \bigoplus_{\mathfrak{p} \in S_p} \mathbb{Z}_p[\![H/H_{\mathfrak{p}}]\!] \to \mathbb{Z}_p \to 0.$$

Remark

If E is CM, then $X = X'$, and by hypothesis $\mathbb{Z}_p[\![H/H_{\mathfrak{p}}]\!]^{\psi} = 0$, so $X^{\psi} \cong (H^2_{\mathrm{Iw}})^{\psi}$.

Notation (inertia groups)

- For $\mathfrak{p} \in S_p$, let $I_{\mathfrak{p}}$ be the inverse limit of direct sums at primes over $\mathfrak{p}$ of inertia groups of the max. abelian pro-p extensions of number fields in K.
- For $P \subset S_p$, we set $I_P = \bigoplus_{\mathfrak{p} \in P} I_{\mathfrak{p}}$.

Remarks

- $I_{\mathfrak{p}}^{\psi}$ has Λ-rank $[E_{\mathfrak{p}} : \mathbb{Q}_p]$
- $I_{\mathfrak{p}}^{\psi}$ is free over Λ by assumption on ψ

Sketch of proof of Theorem

On ψ-eigenspaces, the proposition gives an exact sequence

$$0 \to E^1(X^{\omega\psi^{-1}})(1) \to \mathfrak{X}^\psi \to (\mathfrak{X}^\psi)^{**} \to E^2(X^{\omega\psi^{-1}})(1) \to 0.$$

Since $X^{\omega\psi^{-1}}$ is pseudo-null, the first term vanishes. As $\mathfrak{X}^\psi$ is rank one, $(\mathfrak{X}^\psi)^{**}$ is reflexive of rank one, hence free. The canonical map $(I_{\bar{\mathfrak{p}}}^\psi)^{**} \to (\mathfrak{X}^\psi)^{**}$ is then of free rank one modules, so given up to a choice of basis by some $f \in \Lambda$. Thus, we have a commutative diagram

$$\begin{tikzcd}
 & I_{\bar{\mathfrak{p}}}^\psi \arrow[r, "\sim"] \arrow[d] & (I_{\bar{\mathfrak{p}}}^{**})^\psi \arrow[d] & & \\
0 \arrow[r] & \mathfrak{X}^\psi \arrow[r] \arrow[d, two heads] & (\mathfrak{X}^\psi)^{**} \arrow[r] \arrow[d, two heads] & E^2(X^{\omega\psi^{-1}})(1) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X_{\mathfrak{p}}^\psi \arrow[r] & \Lambda/(f) \arrow[r] & E^2(X^{\omega\psi^{-1}})(1) \arrow[r] & 0.
\end{tikzcd}$$

Since $c_1(X_{\mathfrak{p}}^\psi) = (\mathcal{L}_{\mathfrak{p},\psi})$, we have $(f) = (\mathcal{L}_{\mathfrak{p},\psi})$. Now taking the quotients instead by $I_{\mathfrak{p}}^\psi \oplus I_{\bar{\mathfrak{p}}}^\psi$ and its double dual, we obtain the result.

Issues with generalization

Remark

In the proof, we used the following assumptions and facts:

- Greenberg's conjecture that X^ψ is pseudo-null
- $\mathfrak{X}^\psi$ has rank one, so $(\mathfrak{X}^\psi)^{**}$ is free
- $(I_{\bar{\mathfrak{p}}}^\psi)^{**} \to (\mathfrak{X}^\psi)^{**}$ is a map of free rank one modules, so the cokernel will be cyclic, determined by a p-adic L-function

Remark

For higher degree CM fields, we might replace $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ by a CM type $\mathcal{S}$ and its complement $\bar{\mathcal{S}}$. We face the following issues.

- Greenberg's conjecture is not known to hold even for E imaginary quadratic.
- $\mathfrak{X}^\psi$ has rank d, and $(\mathfrak{X}^\psi)^{**}$ is reflexive but need not be free
- Even if the latter module were free, $(I_{\bar{\mathcal{S}}}^\psi)^{**} \to (\mathfrak{X}^\psi)^{**}$ is given by a matrix on bases, not a single element.

Remark

We resolve these issues as follows:

- Instead of using c_n, which is only defined on codimension n modules, we define t_n of a finitely generated Λ-module to be c_n of its maximal codimension n submodule.
- If Σ properly contains a CM type, then X_Σ^ψ has rank $\ell \geq 1$, and then consider the Λ-exterior power $\bigwedge^\ell X_\Sigma^\psi$ to get a rank one module.
- If we first localize X_Σ^ψ at a prime $\mathfrak{q}$ of Λ of codimension 2, then $(X_{\Sigma,\mathfrak{q}}^\psi)^{**}$ will be reflexive over a regular local ring of Krull dimension 2, hence free.
- For each CM type $\mathcal{S}$ contained in Σ, the module $I_{\Sigma-\mathcal{S}}^\psi$ is free of rank ℓ, so $\bigwedge^\ell I_{\Sigma-\mathcal{S}}^\psi$ is free of rank one. The map

  $$\bigwedge^\ell (I_{\Sigma-\mathcal{S},\mathfrak{q}}^\psi)^{**} \to \bigwedge^\ell (X_{\Sigma,\mathfrak{q}}^\psi)^{**}$$

  is then of free rank one Λ-modules, and the map

  $$\bigwedge^\ell I_{\Sigma-\mathcal{S},\mathfrak{q}}^\psi \to \bigwedge^\ell X_{\Sigma,\mathfrak{q}}^\psi$$

  of rank one modules has cokernel Z with $c_1(Z) = (\mathcal{L}_{\mathcal{S},\psi})$.

Given any two p-adic L-functions attached to distinct CM types, we have a quotient of exterior products corresponding to the pair in the following sense.

Theorem A (BCGKST)

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be distinct CM types, let $\Sigma = \mathcal{S}_1 \cup \mathcal{S}_2$, let $\mathcal{T}_i = \Sigma - \mathcal{S}_i$ for $i \in \{1, 2\}$, and let $\Sigma^c = S_p - \Sigma$. We have an equality

$$t_2\left(\frac{\Lambda}{(\mathcal{L}_{\mathcal{S}_1}, \mathcal{L}_{\mathcal{S}_2})}\right) = t_2\left(\frac{\left(\bigwedge^\ell X_\Sigma^\psi\right)'}{\bigwedge^\ell I_{\mathcal{T}_1}^\psi + \bigwedge^\ell I_{\mathcal{T}_2}^\psi}\right) + t_2\left(\frac{\theta}{\theta_0}\frac{\Lambda}{\mathrm{Fitt}(E^2(X_{\Sigma^c}^{\omega\psi^{-1}})(1))}\right),$$

where $'$ denotes maximal torsion-free quotient, θ is a GCD of $\mathcal{L}_{\mathcal{S}_1,\psi}$ and $\mathcal{L}_{\mathcal{S}_2,\psi}$, and θ_0 is a generator of $t_1(\bigwedge^\ell X_\Sigma^\psi)$.

Remark

Even if X_Σ^ψ is torsion-free, its top exterior power may not be. The reflected Iwasawa module appears in the last term, and θ and θ_0 account for the possible failure of pseudo-nullity.

The proof uses a generalization of the exact sequence relating $\mathfrak{X}$ and $\mathfrak{X}^{**}$.

We have the following analogous result to the imaginary quadratic case in the case that we have two CM types $\mathcal{S}_1$ and $\mathcal{S}_2$ that differ in a degree one prime and its conjugate. In this case, $X^{\psi}_{\mathcal{S}_1 \cup \mathcal{S}_2}$ has Λ-rank one.

Theorem B (BCGKST)

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be distinct CM types with $(\mathcal{S}_1 \cup \mathcal{S}_2) - \mathcal{S}_1 = \{\mathfrak{p}\}$ for some $\mathfrak{p}$ of degree 1 over $\mathbb{Q}$. Let $\bar{\mathcal{S}}_i$ be the complex conjugate type to $\mathcal{S}_i$ for $i \in \{1,2\}$. Assume that $\mathcal{L}_{\mathcal{S}_1,\psi}$ and $\mathcal{L}_{\mathcal{S}_2,\psi}$ are relatively prime. Then

$$c_2\left(\frac{\Lambda}{(\mathcal{L}_{\mathcal{S}_1,\psi}, \mathcal{L}_{\mathcal{S}_2,\psi})}\right) = c_2(X^{\psi}_{\mathcal{S}_1 \cap \mathcal{S}_2}) + c_2((X^{\omega\psi^{-1}}_{\bar{\mathcal{S}}_1 \cap \bar{\mathcal{S}}_2})^{\iota}(1)).$$

We can get an exact isomorphism between the "exterior quotient" and the quotient of the Iwasawa algebra by the corresponding characteristic elements so long as we avoid the support of the "Tate dual of the reflected module".

Theorem C (BCGKST)

Let Σ denote a subset of S_p containing a CM type. Let $\mathcal{S}_1, \ldots, \mathcal{S}_n$ be distinct CM types contained in Σ. Set $\mathcal{T}_i = \Sigma - \mathcal{S}_i$ for each i and $\Sigma^c = S_p - \Sigma$.

Let $\mathfrak{q}$ be a prime of Λ not in the support of $(X_{\Sigma^c}^{\omega\psi^{-1}})^{\iota}(1)$.

- *The $\Lambda_{\mathfrak{q}}$-module $X_{\Sigma,\mathfrak{q}}^{\psi}$ is free of rank ℓ.*
- *We have a canonical isomorphism of $\Lambda_{\mathfrak{q}}$-modules*

$$\frac{\bigwedge^{\ell} X_{\Sigma,\mathfrak{q}}^{\psi}}{\bigwedge^{\ell} I_{\mathcal{T}_1,\mathfrak{q}}^{\psi} + \cdots + \bigwedge^{\ell} I_{\mathcal{T}_n,\mathfrak{q}}^{\psi}} \cong \frac{\Lambda_{\mathfrak{q}}}{(\mathcal{L}_{\mathcal{S}_1,\psi}, \ldots, \mathcal{L}_{\mathcal{S}_n,\psi})}.$$

Remark

The set Σ^c is smaller than a CM type if $n \geq 2$, so $X_{\Sigma^c}^{\omega\psi^{-1}}$ is likely to have high codimension, and is most frequently zero.