Two (Actually One) Applications of a New Basis Theorem for Perfect Sets

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My goal in this talk: Present a new basis theorem for perfect sets* and convince you it is useful.

The theorem is kind of technical looking, so I will start by explaining an application.

*Which arguably shouldn't be called a basis theorem

Application: Embedding partial orders in the Turing degrees

Embedding partial orders into the Turing degrees

Question: What partial orders can you embed into the Turing degrees?

Two obvious restrictions:

- Turing degrees have size continuum
- Each Turing degree has countably many predecessors

Are these the only restrictions?

Question (Sacks): Does every locally countable partial order of size continuum embed into the Turing degrees?

locally countable = every element has at most countably many predecessors

Sacks' question: What's known

Question (Sacks): Does every locally countable partial order of size continuum embed into the Turing degrees?

What's known?

- Provable in ZFC + CH (because it holds in ZFC for partial orders of size ω_1)
- Independent of ZF
- Open: Independent of ZFC?

Theorem (Higuchi): Provable in ZFC for partial orders of height two

Sacks' question: New results

Theorem (Higuchi): It is provable in ZFC that every size continuum, locally countable partial order of height two embeds into the Turing degrees.

This theorem is very robust:

Thoerem (Higuchi-L.): Every locally countable Borel partial order of height two has a Borel embedding into the Turing degrees

However...

Theorem (Higuchi-L.): There is a locally countable Borel partial order of height three that has no Borel embedding into the Turing degrees

How do these results work?

Embedding height two partial orders: Use standard tools from computability theory

Borel embedding height two partial orders: Careful check of ZFC proof + Borel uniformization theorem (Lusin-Novikov)

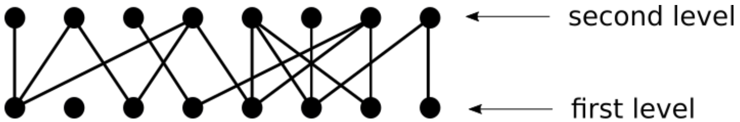
No Borel embedding of height three partial orders: Perfect set theorem for Σ^1_1 sets + new basis theorem for perfect sets

Proof of third result also shows why proof of first result is hard to extend

Embedding height two partial orders

Definition: A partial order has height two if the longest chain has length two

Two levels: elements with no predecessor, and elements with at least one predecessor



Main idea for embedding: Map first level to a set of mutually generic reals and map each element of the second level to a generically chosen upper bound for the images of its predecessors

Embedding height two partial orders

Main idea for embedding: Map first level to a set of mutually generic reals and map each element of the second level to a generically chosen upper bound for the images of its predecessors

For the first level

Lemma 1: There is a perfect set of reals, P, such that no finite set of reals in P computes any other real in P

For the second level

Lemma 2: If A is a countable subset of P then there is a real which computes every element of A and no other element of P

Borel embedding height two partial orders

Thoerem (Higuchi-L.): Every locally countable Borel partial order of height two has a Borel embedding into the Turing degrees

What exactly does this mean?

If $\preceq$ is a locally countable partial order of height two on 2^ω,which is a Borel subset of $2^\omega \times 2^\omega$,then there is a Borel function $f\colon 2^\omega \to 2^\omega$ such that

$$x \preceq y \iff f(x) \leq_T f(y).$$

Borel embedding height two partial orders

How do we modify the proof for embedding partial orders of height two to get the Borel version?

There's really only one problem with the old proof.

Lemma 2: If A is a countable subset of P then there is a real which computes every element of A and no other element of P

This lemma only gives a Borel definition if we have a Borel enumeration of the elements of A. We need a Borel enumeration of the predecessors of each element of the second level of the partial order.

Use a Borel uniformization theorem (Lusin-Novikov)

Nonembedding for height three partial orders

Theorem (Higuchi-L.): There is a locally countable Borel partial order of height three that has no Borel embedding into the Turing degrees

How does this work?

Idea: Let $\preceq$ be a partial order of height three and suppose it has a Borel embedding into the Turing degrees. By the perfect set theorem for Σ^1_1 sets, the image of the first level of $\preceq$ contains a perfect set. We then use this to prove that the image of $\preceq$ has more structure than $\preceq$ itself. Key ingredient: new basis theorem for perfect sets.

A new basis theorem for perfect sets

The new basis theorem

Theorem (L.): If P is a perfect set, A is a countable dense subset, and $a \in 2^\omega$ computes every element of A, then for every x there are $b_1, b_2, b_3, b_4 \in P$ such that $a \oplus b_1 \oplus b_2 \oplus b_3 \oplus b_4 \geq_T x$.

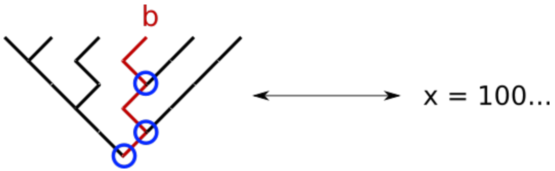
This looks kind of technical. And maybe shouldn't even be called a basis theorem.

Let me back up a bit

A trivial basis theorem

Fact: If $P \subseteq 2^\omega$ is a perfect set and T is a tree such that $P = [T]$ then for every $x \in 2^\omega$ there is some $b \in P$ such that $T \oplus b \geq_T x$.

Proof: Using T we can compute a homeomorphism between 2^ω and P and just take the image of x under this homeomorphism. Essentially, b is a path through T and the bits of x are coded into whether b goes left or right each time T branches.



A slightly less trivial basis theorem

Fact: If $P \subseteq 2^\omega$ is a perfect set and $\langle a_n \rangle_{n \in \mathbb{N}}$ is a countable dense subset of P then for every $x \in 2^\omega$ there is $b \in P$ such that

$$\bigoplus a_n \oplus b \geq_T x$$

Problem: We cannot computably reconstruct T from just a countable dense subset of P.

Solution: We *can* computably find a perfect subtree of T though, which is good enough.

An old basis theorem

Theorem (Groszek-Slaman): If $P \subseteq 2^\omega$ is a perfect set and $\langle a_n \rangle_{n \in \mathbb{N}} \subseteq 2^\omega$ contains a countable dense subset of P, then for every $x \in 2^\omega$ there are $b_1, b_2, b_3 \in P$ such that

$$\bigoplus_{n \in \mathbb{N}} a_n \oplus b_1 \oplus b_2 \oplus b_3 \geq_T x$$

Challenge: We need to avoid getting confused by a_n's which are not actually in P

Solution: Use b_2 and b_3 to provide extra information to help the decoding procedure avoid these a_n's

A new basis theorem

Theorem (L.): If P is a perfect set, A is a countable dense subset, and $a \in 2^\omega$ computes every element of A, then for every x there are $b_1, b_2, b_3, b_4 \in P$ such that $a \oplus b_1 \oplus b_2 \oplus b_3 \oplus b_4 \geq_T x$.

Challenge: The list of functions computable from the real a contains a countable dense subset of P, but it also contains partial functions.

Solution: Use b_2, b_3, b_4 to provide even more information to the decoding procedure to help avoid computations which are not total.

Back to embedding partial orders into the Turing degrees

Applying the basis theorem

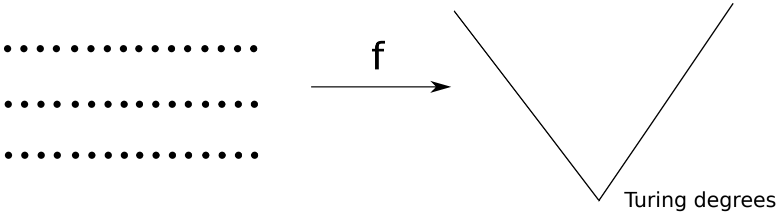
Reminder:

Theorem (Higuchi-L.): There is a locally countable Borel partial order of height three that has no Borel embedding into the Turing degrees

Theorem: Every uncountable Σ^1_1 subset of 2^ω contains a perfect set.

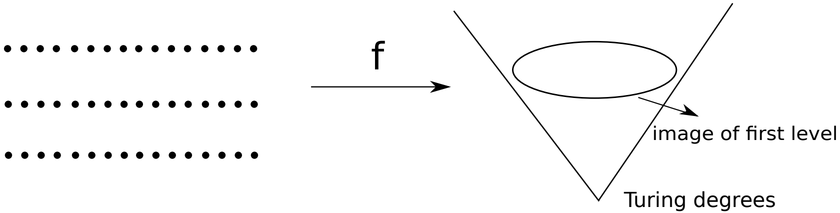
Applying the basis theorem

Let f be a Borel embedding from a sufficiently complicated height three Borel partial order, $\preceq$.



Applying the basis theorem

Since the image of the first level of $\preceq$ is an uncountable Σ^1_1 set, it contains a perfect set, P.



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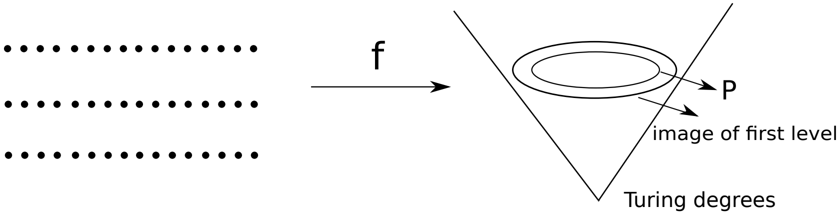
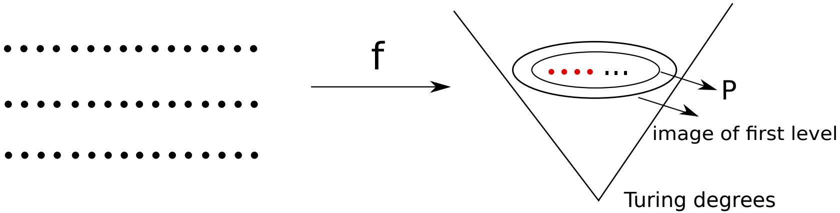


Applying the basis theorem

Let A be a subset of the first level which maps to a countable dense subset of P.



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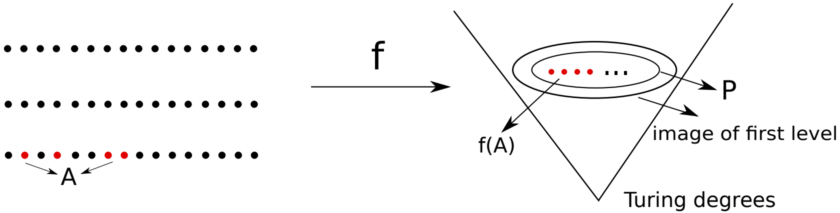
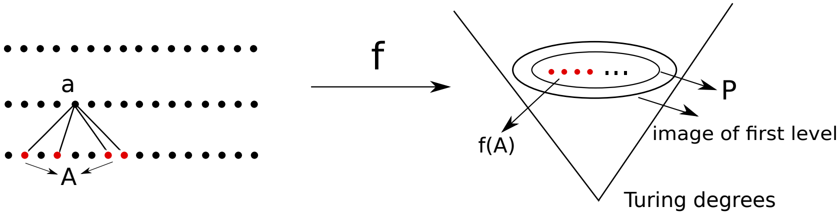


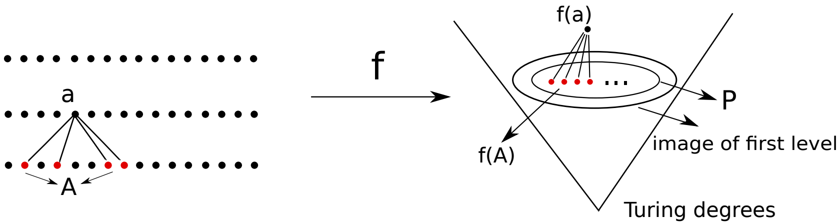
Applying the basis theorem

Let a be an element of the second level which is above every element of A.



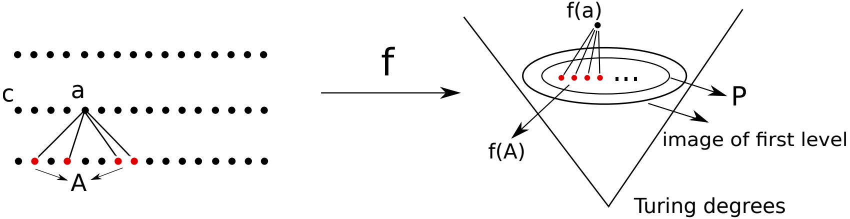
Applying the basis theorem

Let a be an element of the second level which is above every element of A. $f(a)$ computes everything in $f(A)$.



Applying the basis theorem

Now let c be any element of the second level other than a.



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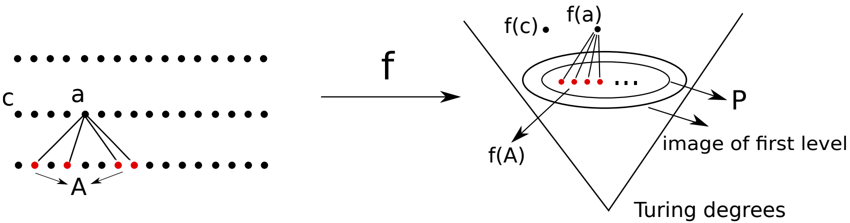
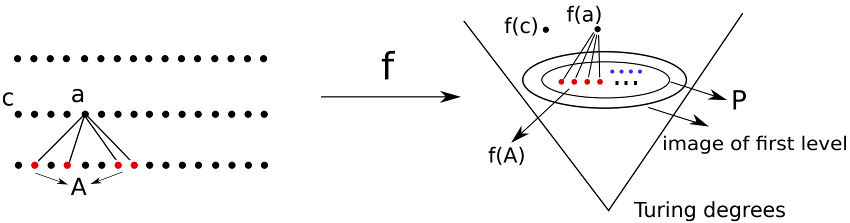


Applying the basis theorem

By the basis theorem, we can pick b_1, b_2, b_3, b_4 in the first level such that $f(a) \oplus f(b_1) \oplus \ldots \oplus f(b_4) \geq_T f(c)$.



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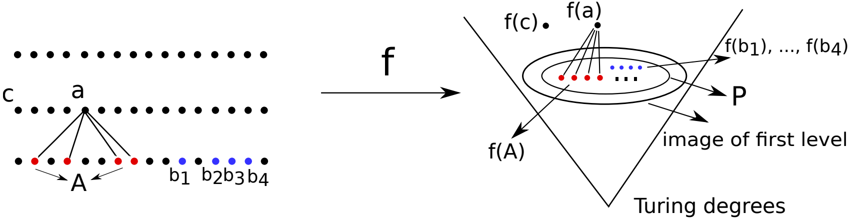
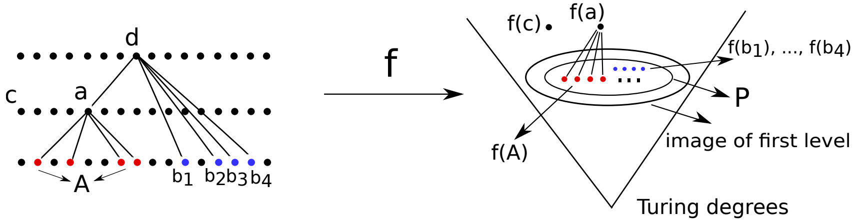


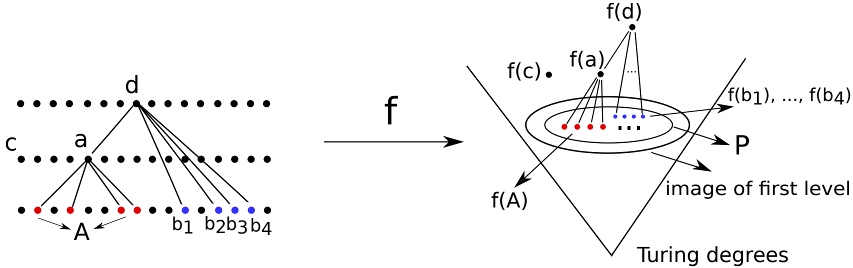
Applying the basis theorem

Let d be an element of the third level which is above a, b_1, b_2, b_3, b_4 but not above c.



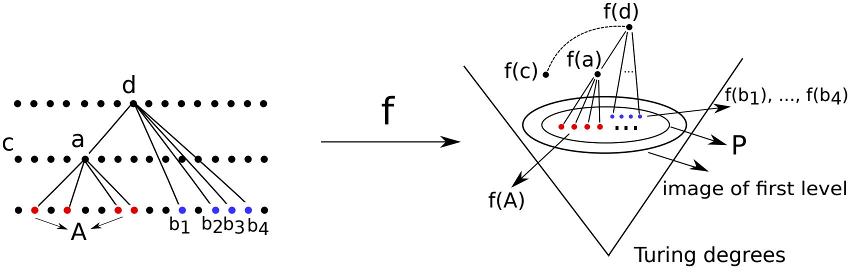
Applying the basis theorem

Then $f(d) \geq_T f(a) \oplus f(b_1) \oplus \ldots \oplus f(b_4)$ so $f(d)$ computes $f(c)$.



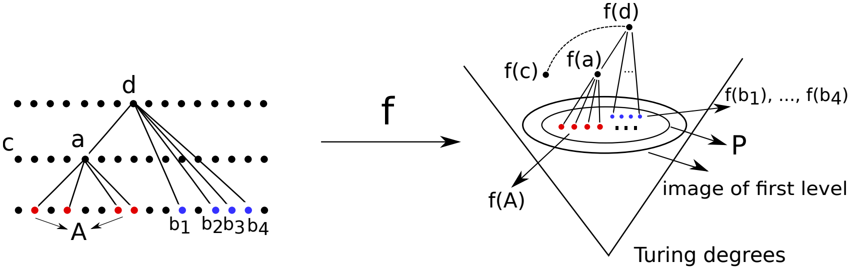
Applying the basis theorem

Then $f(d) \geq_T f(a) \oplus f(b_1) \oplus \ldots \oplus f(b_4)$ so $f(d)$ computes $f(c)$.



Applying the basis theorem

Then $f(d) \geq_T f(a) \oplus f(b_1) \oplus \ldots \oplus f(b_4)$ so $f(d)$ computes $f(c)$, a contradiction since d is not above c.



Finishing the proof: a technicality

We made several assumptions about $\preceq$ in the proof.

- The first level of $\preceq$ is uncountable
- Every countable subset of the first level has an upper bound on the second level.
- For every finite subset S of the first two levels and any c in the second level which is not in S, there is an element in the third level above everything in S but not above c.

It is not hard to show that there is a locally countable Borel partial order with these properties.

A comment on the proof

The only time we used anything about the partial order and the embedding being Borel was to show that the image of the first level of the partial order contained a perfect set.

Therefore, if we try to embed an arbitrary height three partial order and end up with a perfect set in the image of the first level, then we are going to fail.

But this is exactly what the proof for the height two case did!So the proof we just gave shows why that method cannot be extended much further.

A silly question?

We have explained why it seems hard to show that all locally countable partial orders of height three can be embedded into the Turing degrees. Could it be hard because it's not provable in ZFC?

Question: Is it consistent with ZFC that there is a height three, locally countable partial order of size continuum that does not embed into the Turing degrees?

This seems crazy to me, but it is surprisingly hard to try to build an embedding even for partial orders of height three.

Other applications

Benny Siskind and I have also used the basis theorem to help prove part 1 of Martin's conjecture for order preserving functions on the Turing degrees.

Key idea: use the basis theorem to show that if f is an order preserving Borel function on the Turing degrees then for every x, there is some y such that $f(y) \geq_T x$.

The proof is very similar to the one given above. First show that the range of f contains a perfect setand then apply the basis theorem.

Are there other places where this tool can be applied?