A NOTE ON A QUESTION OF SACKS: IT IS HARDER TO EMBED HEIGHT THREE PARTIAL ORDERS THAN HEIGHT TWO PARTIAL ORDERS

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Abstract. A long-open question of Sacks asks whether it is provable in ZFC that every locally countable partial order of size continuum embeds into the Turing degrees. We show that this holds for partial orders of height two, but provide evidence that it is hard to extend this result even to partial orders of height three. In particular, we show that the result for height two partial orders works both in the theory $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ and in the Borel context (where the partial orders and embeddings are required to be Borel), but that the analogous result for height three partial orders is false in both of those settings. We also consider how our results fit into the theory of locally countable Borel quasi orders.

1. Introduction

A perennial question in computability theory is to determine which structures can be embedded into the Turing degrees. When the structures under consideration are partial orders, there are two obvious restrictions: the partial order of Turing degrees has size continuum and every Turing degree has at most countably many predecessors. Thus any partial order which embeds into the Turing degrees must have size at most continuum and all of its elements must have at most countably many predecessors. A long-open question of Sacks asks whether these are the only restrictions.

More precisely, if every element of a partial order has at most countably many predecessors then we say that the partial order is **locally countable**. In 1963, Sacks asked whether every locally countable partial order of size continuum can be embedded into the Turing degrees [Sac63]. Sacks himself proved that this holds in $\mathsf{ZFC} + \mathsf{CH}$ (because it holds in ZFC for all locally countable partial orders of size ω_1) and it is known to be independent of ZF. But whether it is independent of ZFC is an open question.

We will not resolve that question in this paper. Instead, we will present a curious phenomenon related to it. First, we will show (in ZFC) that every size continuum, locally countable partial order *of height two* can be embedded into the Turing degrees. Superficially it appears plausible that this proof could be generalized substantially, but we will provide evidence that it is hard to extend even to partial orders of height three. Here's what we mean. We will show that the result for partial orders of height two is fairly robust: it works not only in ZFC but also in the theory $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ ($\mathsf{AD}_{\mathbb{R}}$ is an axiom of set theory which is inconsistent with the axiom of choice; we will not assume that the reader is familiar with it and we will explain in the body of the paper everything about it that we need). However, we will show that the corresponding statement for partial orders of height three fails—it is provable in $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ that there is a height three, locally countable partial order of size continuum that does not embed into the Turing degrees.

There are two main ingredients in proving the nonembeddability result for height three partial orders. The first is the perfect set theorem, which says that in $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$, every uncountable subset of the reals contains a perfect set; this is the only use we will make of $\mathsf{AD}_{\mathbb{R}}$

in the nonembeddability proof. The second is a basis theorem for perfect sets first proved in [LS21].

It turns out that this nonembeddability result also has consequences for embedding partial orders into the Turing degrees in ZFC. We will see that our proof actually shows that the technique we used for the height two case cannot be extended to work for height three partial orders, even in ZFC. More precisely, the embedding we construct for a height two partial order embeds the first level of the partial order as (the Turing degrees of) a perfect set of reals. But we will see that our nonembeddability proof for height three partial orders shows that there is a certain height three partial order so that no embedding of this partial order into the Turing degrees can embed the first level as a set of Turing degrees which contains a perfect set of reals (our use of $\mathsf{AD}_{\mathbb{R}}$ in the proof is limited to showing that every embedding must do this). This complements another obstacle to embedding partial orders into the Turing degrees in ZFC discovered by Groszek and Slaman [GS83] (and extended by Kumar in an unpublished note [Kum19]). We will discuss this more in section 5.

We will also show that our results have parallels in the world of Borel partial orders. It is provable in ZF that any height two, locally countable Borel partial order on the reals has a Borel embedding into Turing reducibility (considered as a Borel relation on the reals) and also that there is a height three, locally countable Borel partial order on the reals which has no Borel embedding into Turing reducibility. The proofs are very similar to the $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ proofs. For example, we can replace the perfect set theorem provable from $\mathsf{AD}_{\mathbb{R}}$ with a theorem of descriptive set theory (provable in ZF), which says that every uncountable $\mathbf{\Sigma}^1_1$ set of reals contains a perfect subset.

We finish the paper by pointing out a connection between our results and the theory of countable Borel equivalence relations. Kechris has conjectured that the Turing degrees are a universal countable Borel equivalence relation. Our results show that they are not a universal locally countable Borel quasi order (this was also proved by related means in [LS21]). We will explain this more fully in the last section of the paper.

For an introduction to past results about embedding various partial orders and related structures into the Turing degrees, see the survey by Montalbán [Mon09]. For an introduction to the theories of countable Borel equivalence relations and locally countable Borel quasi orders, see the upcoming survey by Kechris (currently available as a preprint) [Kec21].

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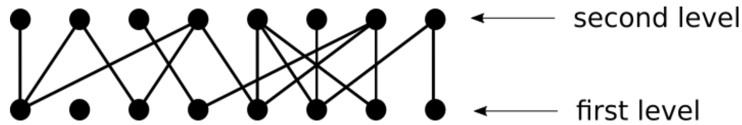
2. Embedding Height Two Partial Orders: ZFC Case

In this section we will explain how to embed any height two, locally countable partial order of size continuum into the Turing degrees. For now, we will just prove this theorem in ZFC. In the next section we will see how to refine the proof to work in $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ and in section 6 we will see how to get it to work in the Borel setting.

Before we dive into the details of the proof, let's discuss the strategy. Recall that a partial order of height two is a partial order with no chains of length greater than two. We can think of such a partial order as consisting of two "levels": a first level which contains all elements with no predecessors and a second level which contains those elements which have one or more predecessors. Note that the elements of a single level are all incomparable with each other. Here's a picture.



The basic idea of the proof is that we will map the elements of the first level to a bunch of mutually generic reals (or, rather, the Turing degrees of these reals) and map each element of the second level to a sufficiently generic upper bound of the images of its predecessors.

There is one wrinkle in the proof: we need to ensure that even when two elements of the second level have exactly the same predecessors, they get mapped to incomparable Turing degrees[1]. One way to handle this is to insert a unique point below each element of the second level, which is not below any other elements of the partial order. This ensures that the elements of the second level all have distinct sets of predecessors and an embedding of this new partial order yields an embedding of the original partial order by forgetting about the new elements that we added. When we give the proof in detail, we will not quite explain things in this way, but it is essentially what will happen.

We will break the proof into two lemmas, the first of which tells us that we can find mutually generic reals to map the elements of the first level to and the second of which tells us that we can always find sufficiently generic upper bounds to map the elements of the second level to. Both lemmas are essentially folklore, though we are not aware of anywhere that they are written up in the precise form we would like to use.

To state these two lemmas, we need to recall some definitions. First, we will need the definition of a "perfect tree."

Definition 2.1. *A tree $T \subseteq 2^{<\omega}$ is called a* ***perfect tree*** *if every node in T has incomparable descendants in T.*

The name "perfect tree" is used because the set of infinite paths through T is a perfect subset of 2^ω. You can picture a perfect tree T as a kind of warped version of $2^{<\omega}$: there are no dead ends and if you follow any path long enough you will eventually come to a place where you can choose to go left or right and remain in the tree either way. In $2^{<\omega}$ you can make this decision after every step; in a perfect tree you may have to take many steps before being able to make this decision. Just as the set of paths through $2^{<\omega}$ is exactly 2^ω, the set of paths through a perfect tree is always homeomorphic to 2^ω.

Notation 2.2. If $T \subseteq 2^{<\omega}$ is a tree then $[T]$ denotes the set of infinite paths through T (which is a subset of 2^ω).

Fact 2.3. *If $T \subseteq 2^{<\omega}$ is a perfect tree then $[T]$ is homeomorphic to 2^ω and therefore has size continuum.*

The idea is that we will embed the first level as the set of paths through some carefully chosen perfect set of reals. Essentially we want to pick a perfect tree whose paths are all mutually generic, but for simplicity we will only require that the paths form a **Turing independent set**, which will be sufficient for our purposes here.

Definition 2.4. *A set $X \subset 2^\omega$ of reals is called a* ***Turing independent set*** *if no finite subset of X computes any other element of X—i.e. if $a_0, \ldots, a_n \in X$ and b is any element of X not equal to any a_i then $a_0 \oplus \ldots \oplus a_n$ does not compute b.*

[1] A similar problem arises if an element of the second level has exactly one predecessor: we need to make sure it gets sent to a different Turing degree than its predecessor.

We can now state our first lemma, which gives us a Turing independent set of size continuum. The argument is due to Sacks in his paper [Sac61], though he did not state it in terms of perfect trees.

Lemma 2.5 (Sacks [Sac61] Theorem 3)**.** *There is a perfect tree T such that $[T]$ is a Turing independent set.*

Proof. First consider how one might construct a perfect tree none of whose branches is computable. To do this, we need to ensure that every branch of the tree disagrees with each total computable function in at least one place. We can accomplish this by "growing" the tree from the root node up in a series of stages. At each stage we have built a finite tree and we continue growing it by extending the leaf nodes (i.e. by adding children to the leaf nodes, children to those children, and so on). On alternate stages we can add incomparable children below every leaf node (to make sure the tree is perfect) and extend each leaf node to make sure any branch which extends it disagrees with the next total computable function (note that there is no need to make the tree itself computable).

To make sure that no finite set of branches computes any other branch, we can do something similar but now instead of extending leaf nodes one at a time to make them disagree with the next computable function, we need to extend finite sets of leaf nodes at the same time to make sure the next computable function which uses those branches as an oracle disagrees with the branches of the tree which extend the other leaf nodes.

We will now describe this a bit more formally. We will form a sequence of finite subtrees of $2^{<\omega}$, $T_0 \subset T_1 \subset T_2 \subset \ldots$ such that T_{n+1} is an end extension of T_n (every node in $T_{n+1} \setminus T_n$ extends a leaf node of T_n). The final tree will be obtained as $T = \bigcup_{n\in\mathbb{N}} T_n$. We can start with T_0 as the tree just consisting of a single root node and nothing else (i.e. just the empty sequence).

Now we will explain how to extend T_n to T_{n+1}. The idea, again, is to first split every leaf node in T_n and then extend all of them without splitting in order to make sure that no finite subset of them is correctly computing any of the others using the the n^{th} Turing functional. To this end, first let T_n^0 be the tree formed by adding incomparable children below each leaf node of T_n. In other words,

$$T_n^0 = T_n \cup \{\sigma^\frown 0 \mid \sigma \text{ is a leaf node of } T_n\} \cup \{\sigma^\frown 1 \mid \sigma \text{ is a leaf node of } T_n\}.$$

Next, let Φ be the n^{th} Turing functional in some standard enumeration and let $\text{Leaves}(T_n^0)$ be the set of leaves of T_n^0. We will form a series of end extensions $T_n^0 \subset T_n^1 \subset \ldots \subset T_n^k$ (where k is the number of nonempty subset of $\text{Leaves}(T_n^0)$) and take $T_{n+1} = T_n^k$. In each of these extensions, we will not split any nodes. In other words, each leaf of T_n^0 will have at most one descendant at each level of T_n^i.

Suppose we have already formed T_n^i and let S be the i^{th} nonempty finite subset of $\text{Leaves}(T_n^0)$. We will now explain how to form T_n^{i+1}. Our goal is to ensure that no set of branches extending the nodes in S can compute any branch extending any other leaf node of T_n^0.

Since we never split any nodes in any of the previous extensions of T_n^0, each element of S corresponds to a unique leaf of T_n^i. Let $\sigma_1, \ldots, \sigma_l$ denote these leaves of T_n^i. Let N be a number larger than the height of T_n^i. Now either we can find extensions $\tau_1, \ldots, \tau_l$ of $\sigma_1, \ldots, \sigma_l$ such that $\Phi^{\tau_1 \oplus \ldots \oplus \tau_l}(N)$ converges or we can't find such extensions. In the former case, define T_n^{i+1} from T_n^i by extending each σ_j to τ_j and extending all other leaf nodes of T_n^i to strings whose N^{th} bit disagrees with $\Phi^{\tau_1 \oplus \ldots \oplus \tau_l}(N)$. In the latter case, set $T_n^{i+1} = T_n^i$.

Now let's check that $[T]$ is really a Turing independent set. Let $x_1, \ldots, x_l$ and y be distinct elements of $[T]$ and let Φ be any Turing functional. Let n be some number large enough that all of $x_1, \dots, x_l$ and y correspond to distinct leaf nodes in T_n and chosen so that the n^{th} Turing functional is equivalent to Φ (we are assuming that every computable function shows up infinitely often in whatever enumeration we are using). Suppose that $x_1, \dots, x_l$ correspond to the i^{th} set of leaves of T_n^0. Then our definition of T_n^{i+1} ensures that either $\Phi^{x_1 \oplus \dots \oplus x_l}$ disagrees with y in at least one place (this corresponds to the first case in our construction above) or $\Phi^{x_1 \oplus \dots \oplus x_l}$ is not a total function (this corresponds to the second case). □

Our second lemma guarantees we can find sufficiently generic upper bounds to map the elements of the second level of our partial order to. The idea of the proof is originally due to Spector, who used it to show that every increasing sequence of Turing degrees has an exact pair of upper bounds (see Theorem 6.5.3 in Soare [Soa16]).

Lemma 2.6. *Suppose T is a perfect tree such that $[T]$ is Turing independent. Then every countable subset of $[T]$ has an upper bound in the Turing degrees which does not compute any other element of $[T]$.*

Proof. Suppose A is a countable subset of $[T]$ and $x_0, x_1, x_2, \dots$ is an enumeration of the elements of A. To keep things simple, we'll say that repetitions are allowed in this enumeration, which allows us to treat the cases where A is finite or infinite in the same way.

Here's the idea of the proof. We will construct an element y of $2^{\omega \times \omega}$ such that column n of y consists of some finite string followed by x_n and this y will be the upper bound we are after. It is easy to see that any such y computes each element of A and so the bulk of the proof consists of showing that if we choose the finite strings in a sufficiently generic way then y does not compute any other element of $[T]$. The proof crucially depends on the fact that $[T]$ is Turing independent.

Formally, we will construct y in a series of stages. At the end of stage n we will have constructed a finite list of finite strings, $\sigma_1, \dots, \sigma_k$ (where k may not be equal to n) and on stage $n+1$ we will add some more strings onto the end of this list. At the end, we will define column i of y to be $\sigma_i \frown x_i$. The idea is that on stage $n+1$ we will ensure that if we run the n^{th} program with oracle y it either does not compute any element $[T]$ or it computes one of $x_1, \dots, x_k$.

We will now explain how to complete one step of this construction. Suppose we have just completed stage n and our list of finite strings is $\sigma_0, \sigma_1, \dots, \sigma_k$. Let us say that a finite string τ in $2^{<\omega \times <\omega}$ (i.e. a finite initial segment of an element of $2^{\omega \times \omega}$) *agrees with y so far* if for each $i \leq k$, column i of τ agrees with σ_i followed by x_i. In other words, τ is a possible initial segment of y given what we have built by the current stage. Let Φ denote the n^{th} Turing functional. We will use the notation Φ^a (where a is a real) to mean the partial function computed by Φ when given oracle a. If τ is a finite string, we will use Φ^τ to mean the partial function computed by Φ when given oracle τ (where the program halts without output when it tries to make a query outside the domain of τ and is allowed to run for at most $|\tau|$ steps on any input). We have four cases to consider.

Case 1: There is some finite string $\tau \in 2^{<\omega \times <\omega}$ which agrees with y so far such that Φ^τ cannot be extended to a path through T. In this case, extend the list $\sigma_0, \sigma_1, \dots, \sigma_k$ to ensure that y is an extension of τ. This guarantees that Φ^y is not in $[T]$.

Case 2: There is some finite string τ which agrees with y so far and some $m \in \mathbb{N}$ such that for every extension τ' of τ which agrees with y so far, $\Phi^{\tau'}$ does not converge on input m. In this case, extend the list $\sigma_0, \sigma_1, \dots, \sigma_k$ to ensure that y is an extension of τ. This guarantees that Φ^y is not total.

Case 3: For every finite string τ which agrees with y so far, Φ^τ is compatible with one of $x_0, x_1, \ldots, x_k$. In this case, do nothing; we are already guaranteed that Φ^y is either not total or is equal to one of $\sigma_0, \sigma_1, \ldots, \sigma_k$.

Case 4: None of the first three cases holds. We claim this case actually cannot happen. Let's explain why.

Because Case 3 does not hold, we can find some finite string τ which agrees with y so far and such that Φ^τ is not compatible with any of $x_0, x_1, \ldots, x_k$. Because Case 2 does not hold, for each m we can find some extension τ' of τ which is agrees with y so far and such that $\Phi(\tau')$ converges on input m; what's more, for any given finite string extending τ which agrees with y so far, we can require τ' to actually extend this string. And because Case 1 does not hold, $\Phi^{\tau'}$ is compatible with some element of $[T]$. We will use this to show that we can use $x_0, x_1, \ldots, x_k$ to compute some element of $[T]$ which is not equal to any of them. This contradicts the fact that $[T]$ is Turing independent.

Let's now describe how to use $x_0, x_1, \ldots, x_k$ to compute another element of $[T]$, thus finishing the proof. Form a sequence $\tau = \tau_0 \prec \tau_1 \prec \tau_2 \prec \ldots$ of finite strings which agree with y so far as follows. Given τ_m, look for some extension τ_{m+1} of τ_m which agrees with y so far and such that $\Phi^{\tau_{m+1}}$ converges on input m. We can always find such a string because Case 2 does not hold.

Let z be the real whose m^{th} bit is equal to $\Phi^{\tau_{m+1}}$ on input m. Note that the first m bits of z are equal to the first m bits of $\Phi^{\tau_{m+1}}$. Because Case 1 does not hold, this means that $z \upharpoonright m$ agrees with some element of $[T]$. Since $[T]$ is closed, this means that z is in $[T]$. And z is not equal to any of $x_0, x_1, \ldots, x_k$ because of our choice of τ.

The final point is that to carry out this whole process, we only need to be able to check when a finite string agrees with y so far. And if we know $x_0, x_1, \ldots, x_k$ then we can compute this, so the whole process is computable in $x_0 \oplus x_1 \oplus \ldots \oplus x_k$. □

We will now explain how to put together the two lemmas to prove the main theorem of this section.

Theorem 2.7. *Every height two, locally countable partial order of size continuum embeds into the Turing degrees.*

Proof. Let $(P, \leq_P)$ be a height two, locally countable partial order of size continuum. We will define a map $\psi \colon P \to 2^\omega$ such that $x \leq_P y$ if and only if $\psi(x) \leq_T \psi(y)$. So even though ψ is a map of P into the reals, it induces an embedding of P into the Turing degrees.

First, let T be a perfect tree such that $[T]$ is Turing independent, as in Lemma 2.5. Since P and $[T]$ both have size continuum, we can find an injective map $f \colon P \to [T]$.

Now, let P_0 be the first level of P and let P_1 be the second level. We will define $\psi(x)$ by cases depending on whether x is in P_0 or P_1. If x is in P_0 then we will simply set $\psi(x) = f(x)$. If x is in P_1 then define $\psi(x)$ as follows. Let $P_{\leq x} = \{y \in P \mid y \leq_P x\}$ be the set of predecessors of x in P; note that $P_{\leq x}$ includes x itself. By Lemma 2.6, we can find a real which computes every element of the image of $P_{\leq x}$ under f but which computes no other elements of $[T]$. Set $\psi(x)$ equal to this real.

Now we need to check that ψ is an embedding. Let x and y be any two distinct elements of P. We need to show that $x \leq_P y$ if and only if $\psi(x) \leq_T \psi(y)$.

First suppose $x \leq_P y$. Notice that x must be in the first level of P and y must be in the second level. Therefore $\psi(x) = f(x)$ and $\psi(y)$ is an upper bound in the Turing degrees for a set which includes $f(x)$, so $\psi(y)$ computes $\psi(x)$.

Now suppose that $x \nleq_P y$. We know that no matter which level x is in, $\psi(x)$ computes $f(x)$. So to show that $\psi(y)$ doesn't compute $\psi(x)$, it is enough to show that $\psi(y)$ doesn't compute

$f(x)$. If y is in the first level of P then this is guaranteed by the fact that $f(x)$ and $f(y)$ are distinct elements of the Turing independent set $[T]$. And if y is in the second level of P then since x is not a predecessor of y, our choice of $\psi(y)$ ensures that it cannot compute $f(x)$. □

The theorem we have just proved is very similar to Theorem 2.2 of the paper "Separating Families and Order Dimension of Turing Degrees" by Kumar and Raghavan [KR21]. That theorem states that a specific height two, locally countable partial order of size continuum—which the authors refer to as $\mathbb{H}_{\mathfrak{c}}$—embeds into the Turing degrees. Obviously this theorem is implied by our Theorem 2.7. But, in fact, it is not too hard to show that every height two, locally countable partial order of size continuum embeds into $\mathbb{H}_{\mathfrak{c}}$ and so the two theorems are actually equivalent.

An especially attentive reader of both papers may note that the proofs of our Theorem 2.7 and Kumar and Raghavan's Theorem 2.2 are somewhat different. Essentially, our proof first embeds the first level of the partial order in "one shot" and then embeds the elements of the second level independently of each other. In contrast, Kumar and Raghavan use transfinite recursion to choose the image of each element of the partial order one at a time in such a way that the image of each element may depend on all the choices already made. Moreover, the elements of the two levels are interleaved in this recursion and so the choices made for elements of the first level may depend on the choices made for elements of the second level.

There are advantages and disadvantages to each approach. Our approach can be extended to show that the result still holds in $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ and in the Borel setting, which is not apparent for Kumar and Raghavan's approach. On the other hand, because our approach embeds the first level of the partial order as a perfect set, it falls prey to the nonembedding result we describe in section 4 and so cannot be extended to handle height three partial orders whereas it is not clear whether this is a problem for the Kumar and Raghavan approach. But on the *other* other hand, the Kumar and Raghavan approach falls prey to a different obstacle to embedding partial orders in the Turing degrees discovered Groszek-Slaman and extended by Kumar [GS83, KR21]. We will have more to say about this in section 5.

3. Embedding Height Two Partial Orders: $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ Case

In this section we will show that the proof from the previous section also works in the theory $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$. This is interesting because, as we will see in the next section, it is provable in $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ that not every height three, locally countable partial order of size continuum embeds into the Turing degrees.

Before we proceed, let's say a few words about the axioms AD and $\mathsf{AD}_{\mathbb{R}}$. The axiom of determinacy (AD for short) is an axiom of set theory which contradicts the axiom of choice but which implies that subsets of the real numbers are very well-behaved. For example, $\mathsf{ZF} + \mathsf{AD}$ proves that every set of real numbers is Lebesgue measurable and has the perfect set property (see Jech, Theorem 33.3 [Jec03]). The axiom $\mathsf{AD}_{\mathbb{R}}$, first introduced by Solovay [Sol78], is a strengthening of AD which implies (among other things) that every binary relation on the reals can be uniformized (i.e. has a choice function).

In fact, the theory $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ is much stronger than necessary to prove the results of this section and the next section. All that is needed beyond ZF to show that height two partial orders can be embedded into the Turing degrees is the statement that every binary relation on the reals can be uniformized. And all that is needed beyond ZF to show that not every height three partial order can be embedded into the Turing degrees is the statement that every set of reals has the perfect set property. On their own, each of these statements is much weaker than $\mathsf{AD}_{\mathbb{R}}$. We chose to state our results in $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ rather than these weaker theories because it is a well-known theory which suffices to prove both results and it demonstrates that the two results are consistent with each other.

So how do we modify the proof of Theorem 2.7 to work in $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$? There is actually only one part of the proof that cannot be carried out in ZF. In Lemma 2.6, we saw how to find an upper bound for any countable subset of a Turing independent perfect set. The proof required choosing a specific enumeration of the elements of the countable subset. In the context of the proof of Theorem 2.7, this means that we had to be able to choose an enumeration of the predecessors of each element of the second level of the partial order that we are trying to embed. The proof still works in $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ because we have access to a sufficiently strong form of choice (the uniformization theorem we mentioned above) to pick these enumerations.

We will now give the details of the proof. We start by stating formally the uniformization theorem provable in $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$.

Fact 3.1 ($\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$; Solovay [Sol78]). *Suppose R is a binary relation on 2^ω. There is a function $f\colon 2^\omega \to 2^\omega$ such that for all $x \in 2^\omega$,*

$$\exists y\, R(x, y) \iff R(x, f(x)).$$

Corollary 3.2 ($\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$). *If R is a binary relation on 2^ω with nonempty, countable sections (i.e. for each $x \in 2^\omega$, the set $\{y \mid R(x, y)\}$ is nonempty and countable) then there is a function $f\colon 2^\omega \to (2^\omega)^\omega$ such that for each $x \in \mathrm{dom}(R)$, $f(x)$ is an enumeration of the section of R at x (where repeats are allowed in the enumeration).*

Proof. Let S be the relation on $2^\omega \times (2^\omega)^\omega$ defined by

$$S(x, y) \iff y \text{ is an enumeration of the set } \{z \mid R(x, z)\}.$$

Because there is a bijection between 2^ω and $(2^\omega)^\omega$, we can think of S as a binary relation on 2^ω. Thus by Fact 3.1, S can be uniformized. Since a uniformization of S is a function that picks an enumeration of the sections of R, we are done. ◻

The theorem now follows from the proof of Theorem 2.7, replacing the one use of choice in that proof with the corollary above.

Theorem 3.3 ($\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$). *Every height two, locally countable partial order of size continuum embeds into the Turing degrees.*

4. (Not) Embedding Height Three Partial Orders: $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ Case

In this section, we will show that it is provable in $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ that there is some height three, locally countable partial order of size continuum which does not embed into the Turing degrees. The only special fact about $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ that we need is the following[2].

Theorem 4.1 ($\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$; [Jec03] Theorem 33.3). *Every subset of 2^ω is either countable or contains a perfect set.*

We will use this theorem in conjunction with the following basis theorem for perfect sets, first proved in [LS21].

Theorem 4.2 (ZF). *Suppose A is a perfect set, B is a countable dense subset of A and x is a real which computes every element of B. Then for any $y \in 2^\omega$ there are $z_0, z_1, z_2, z_3 \in A$ such that $x \oplus z_0 \oplus z_1 \oplus z_2 \oplus z_3 \geq_T y$.*

[2]Which, as we have noted, is actually provable in much weaker theories than $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$, including $\mathsf{ZF} + \mathsf{AD}$.

Our proof proceeds in two steps. First we will show that no height three partial order with certain properties embeds into the Turing degrees and then we will show that there is a height three, locally countable partial order of size continuum which has those properties.

Lemma 4.3 ($\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$)**.** *Suppose $(P, \leq_P)$ is a height three partial order with the following properties.*

(1) *The first level of P (i.e. the set of elements in P which have no predecessors) has size continuum.*
(2) *Every countable subset of the first level of P has an upper bound in the second level.*
(3) *For every finite subset S of the first two levels of P and every x in the second level which is not in S, there is some y in the third level which is above every element of S but not above x.*

Then P cannot be embedded into the Turing degrees.

Proof. Suppose for contradiction that f is an embedding of P into the Turing degrees. Roughly the argument will go as follows: the image of the first level of P under f is uncountable and so it must contain a perfect set. We can then use the basis theorem for perfect sets stated above to conclude something about the image of P in the Turing degrees which contradicts the assumptions we are making about the structure of P.

There is something odd about this sketch that we have to clear up, though: the image of the first level of P under f is a set of *Turing degrees* but the perfect set theorem and the basis theorem for perfect sets are both statements about sets of *reals*. The solution to this discrepancy is to consider not the image of f, but rather the set of reals which are in some Turing degree in the image of f. In the rest of the proof, we will implicitly identify a set of Turing degrees with the set of reals which are contained in those Turing degrees.

As we have already said, the image under f of the first level of P is an uncountable set of reals and hence contains a perfect set (again, what we really mean here is not quite the image of the first level of P under f, but rather the set of reals contained in one of the Turing degrees in this set). Pick a countable dense subset of this perfect set and let A be the elements of the first level of P which map to this countable dense subset. Let x be an element of the second level of P which is an upper bound for A (and thus $f(x)$ computes every element of the countable dense subset).

Now let y be an element of the second level of P which is not equal x[3]. By the basis theorem (Theorem 4.2), there are four elements, a_0, a_1, a_2, a_3, of the perfect set which, together with $f(x)$, compute $f(y)$. Since the perfect set is contained in the image of the first level of P under f, this means there are four elements, w_0, w_1, w_2, w_3, of the first level of P which map to (the Turing degrees of) a_0, a_1, a_2, a_3.

Pick an element z of the third level of P which is above these w_0, w_1, w_2, w_3 and above x, but not above y. Since z is above w_0, w_1, w_2, w_3 and x, $f(z)$ must compute a_0, a_1, a_2, a_3 and $f(x)$ and hence also computes $f(y)$. But since we chose z to not be above y, this contradicts the fact that f is an embedding. □

Lemma 4.4 (ZF)**.** *There is a height three, locally countable partial order of size continuum with the properties listed in the statement of Lemma 4.3.*

Proof. We can essentially take the free height three, locally countable partial order on continuum-many generators. Let's explain what that means.

Let P_0 be a set of size continuum (2^ω, say), let P_1 be the set of countable sequences of elements of P_0 (with repetitions allowed—i.e. $P_1 = P_0^\omega$) and let P_2 be the set of countable

[3]Notice that the properties of P imply the existence of two distinct elements in the second level of P.

sequences of elements of $P_0 \cup P_1$ which have at least one element in P_1. Let $(P, \leq_P)$ be the partial order whose domain is $P_0 \cup P_1 \cup P_2$ and where $x \leq_P y$ if one of the three following conditions holds

- $x = y$
- y is in $P_1 \cup P_2$ and for some $n \in \mathbb{N}$, $y_n = x$ (recall that the elements of P_1 and P_2 are countable sequences)
- y is in $P_1 \cup P_2$ and for some $n, m \in \mathbb{N}$, y_n is in P_1 and $(y_n)_m = x$.

It is easy to check that $(P, \leq_P)$ is a height three, locally countable partial order of size continuum such that the conditions required by Lemma 4.3 hold. □

Together, Lemmas 4.3 and 4.4 imply the main theorem of this section.

Theorem 4.5 ($\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$)**.** *There is a height three, locally countable partial order of size continuum which does not embed into the Turing degrees.*

5. Obstacles to Embedding Partial Orders into the Turing Degrees

Our proof of Theorem 4.5 actually shows something stronger than that not every height three partial order embeds into the Turing degrees in $\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$. It shows that if $(P, \leq_P)$ is the partial order from Lemma 4.4, then any strategy for embedding P into the Turing degrees which ends up with a perfect set contained in the image of the first level of P is guaranteed to fail, even if we are working in ZFC. But our proof that height two partial orders embed into the Turing degrees began by embedding the first level of the partial order as a perfect set! Thus there can be no direct extension of our proof of Theorem 2.7 to work for partial orders of height three. In this section, we will discuss how this observation fits into other known obstacles to embedding partial orders into the Turing degrees.

We will begin our discussion with fairly general considerations. Suppose you want to embed an arbitrary locally countable partial order into the Turing degrees. How might you go about it? A reasonable approach is to pick a well-ordering of the elements of the partial order and proceed by transfinite recursion. In other words, pick up elements of the partial order one at a time and show that as long as you have embedded fewer than continuum many elements so far, there is always a place to send the next element. This is essentially the approach taken by Sacks to embed locally countable partial orders of size ω_1 in [Sac63] and also the approach taken by Kumar and Raghavan to embed height two partial orders [KR21].

A fundamental obstacle to using this approach to embed an arbitrary locally countable partial order was discovered by Groszek and Slaman [GS83]. They showed that it is consistent with ZFC that there is a maximal Turing independent set of size less than continuum. Thus if you want to embed the elements of a partial order into the Turing degrees by a transfinite induction of length continuum, you have to be careful not to end up with this particular Turing independent set in the range of your embedding after fewer than continuum many steps. For suppose you did end up with this set in the range of your embedding. Then if you later encounter another element of the partial order which is sufficiently independent of all the others you have seen so far, there will be nowhere to send it. A potential solution to this is to make the transfinite recursion satisfy some stronger inductive assumption that prevents this situation from occurring, but no suitable condition has been identified so far. In [Kum19], Kumar used Groszek and Slaman's technique to show that even if you are embedding a height three partial order whose first level has size ω_1, a similar obstacle may occur.

From these results of Groszek, Slaman and Kumar, we know that it is tricky to embed locally countable partial orders into the Turing degrees by using transfinite recursion. This suggests that a more structural approach may work better. To make things simple, suppose that we are trying to embed a locally countable partial order of finite height. A reasonable-sounding approach might be to find an especially "nice" subset of the Turing degrees to map the first level to, and then use the features of this "nice" subset to find a "nice" subset of the Turing degrees to map the second level to, and so on. Note that this is exactly the approach we have taken in section 2.

But as we have noted, the results of the previous section show that there are also obstacles to extending this approach. In particular, if by "nice" subset we mean "perfect set" then it is guaranteed to fail for sufficiently complicated partial orders of height at least three. This is made precise in the following theorem, whose proof is just a subset of our proof of Theorem 4.5.

Theorem 5.1. *There is a height three, locally countable partial order of size continuum, $(P, \leq_P)$, such that for any function f from P into the Turing degrees, if the image of f on the first level of P contains a perfect set of reals then f cannot be an embedding.*

It is common to phrase obstacles to embedding partial orders into the Turing degrees in terms of obstacles to extending or modifying Turing independent sets. We can also do that here. The following theorem easily implies Theorem 5.1 and its proof is more or less the same.

Theorem 5.2. *Suppose A is a perfect subset of 2^ω which is Turing independent, B is a countable dense subset of A and x is a real which computes every element of B. Then $(A \setminus B) \cup \{x\}$ is not Turing independent.*

It would be interesting to know if this obstacle could be combined with the obstacles identified by Groszek, Slaman, and Kumar to resolve Sacks' question, but at present we have no idea how to do so. One rather incredible possibility is indicated by the following question.

Question 5.3. *Is it consistent with* ZFC *that there is a height three, locally countable partial order of size continuum which cannot be embedded into the Turing degrees?*

It sounds rather unbelievable that this question could have a positive answer, but it is surprisingly difficult to rule it out.

6. Embedding Height Two and Height Three Partial Orders: Borel Case

The results that we have proved in $\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$ in sections 3 and 4 have analogs in the theory of Borel relations on the real numbers, which we will formulate and prove in this section. In particular, we will show that every height two, locally countable Borel partial order on 2^ω has a Borel embedding into the Turing degrees but that the corresponding statement for height three partial orders is false. Our proof of the latter result is essentially the same as the $\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$ proof (modulo swapping out the perfect set theorem provable in $\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$ for the perfect set theorem for $\mathbf{\Sigma}^1_1$ sets which is provable in ZF). For the former result, the proof is somewhat more involved, though still mostly the same as the $\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$ proof. Before we get into the proofs, though, let's give a precise definition of Borel embedding.

Definition 6.1. *Suppose R and S are Borel subsets of $2^\omega \times 2^\omega$. We say there is a* ***Borel embedding*** *of R into S if there is a Borel function $f \colon 2^\omega \to 2^\omega$ such that for all $x, y \in 2^\omega$,*

$$(x, y) \in R \iff (f(x), f(y)) \in S.$$

6.1. Borel Embedding Height Two Partial Orders.

Suppose that $\leq_P$ is a height two, locally countable partial order on 2^ω, which is Borel (considered as a subset of $2^\omega \times 2^\omega$). We will show that there is a Borel embedding of $\leq_P$ into Turing reducibility, $\leq_T$ (considered also as a subset of $2^\omega \times 2^\omega$).

There are two obstacles to adapting the proof of Theorem 2.7 to the Borel case. The first is an obstacle that we also encountered when adapting the proof to work in $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$—namely that for each element of the second level of the partial order, we need to choose an enumeration of its predecessors. However, we do not have access to the same kind of uniformization result that we used in the $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ case; it is simply not true that every binary Borel relation has a Borel uniformization. However, it turns out that corollary 3.2, which is all that we really needed in the proof, does have a Borel version, called the Lusin-Novikov theorem (which also has applications in the theory of countable Borel equivalence relations).

Theorem 6.2 (Lusin-Novikov uniformization theorem; [Kec95] Theorem 18.10)**.** *Suppose R is a Borel subset of $2^\omega \times 2^\omega$ with countable sections (i.e. for each $x \in 2^\omega$, the set $\{y \mid (x, y) \in R\}$ is countable). Then both of the following hold:*

(1) *The domain of R (i.e. the set $\{x \mid \exists y\, (x, y) \in R\}$) is Borel.*
(2) *R can be written as a countable union of Borel subsets of $2^\omega \times 2^\omega$ which are graphs of partial functions.*

The second obstacle is that in both the ZFC and $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ cases, we defined the embedding $\psi(x)$ by cases depending on whether x was in the first or second level of the partial order. So to ensure that we actually get a Borel embedding, we need to show that the first and second levels of the partial order are both Borel sets. Because the partial order is locally countable, this follows from the first part of the Lusin-Novikov theorem.

We will now proceed with a formal proof. For clarity, we will begin by stating a refined version of Lemma 2.6 (which guaranteed the existence of sufficiently generic upper bounds). The proof just consists of noting that other than choosing an enumeration of the countable set, every part of the construction of the upper bound in Lemma 2.6 was arithmetically definable and thus yields a Borel map from enumerations to upper bounds.

Lemma 6.3. *Suppose T is a perfect tree such that $[T]$ is Turing independent. Then there is a Borel function $g\colon (2^\omega)^\omega \to 2^\omega$ such that for any sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ of elements of $[T]$, $g(\langle x_n\rangle_{n\in\mathbb{N}})$ computes all of the x_n's but does not compute any other element of $[T]$.*

Theorem 6.4. *Every height two, locally countable Borel partial order on 2^ω has a Borel embedding into Turing reducibility.*

Proof. Let $\leq_P$ be a height two, locally countable Borel partial order on 2^ω. Recall that in the proof of Theorem 2.7, we constructed a map $\psi\colon P \to 2^\omega$ such that $x \leq_P y$ if and only if $\psi(x) \leq_T \psi(y)$. In this proof, we will simply give a definition of the function ψ which makes it clear why it is Borel; the proof that it is an embedding is unchanged (and we no longer need to worry that it maps into the reals rather than the Turing degrees, because a Borel embedding into Turing reducibility *is* a map into the reals).

Let T be a perfect tree such that $[T]$ is Turing independent, as in Lemma 2.5. Let $f\colon 2^\omega \to [T]$ be a homeomorphism. By the Lusin-Novikov theorem, there is a Borel function $F\colon 2^\omega \to (2^\omega)^\omega$ such that for each $x \in 2^\omega$, $\langle F(x, n)\rangle_{n\in\mathbb{N}}$ enumerates the set of $\leq_P$-predecessors of x (possibly with duplicates)[4]. By Lemma 6.3, there is a Borel function $g\colon (2^\omega)^\omega \to 2^\omega$, which, when given any sequence of elements of $[T]$, outputs a real which computes all of them but does not

[4]It may seem that there is a slight problem here since the Lusin-Novikov theorem only promises a *partial* function. But by the first part of the Lusin-Novikov theorem, the domain of each $F(-, n)$ is Borel so we can extend F to a total Borel function by setting $F(x, n) = x$ anytime it is undefined.

compute any other element of $[T]$. Now define ψ as follows

$$\psi(x) = \begin{cases} f(x) & \text{if } \forall y \in 2^\omega \, (y \not<_P x) \\ g(n \mapsto f(F(x,n))) & \text{if } \exists y \in 2^\omega \, (y <_P x). \end{cases}$$

By the first part of the Lusin-Novikov theorem, the set $\{x \mid \exists y \, (y <_P x)\}$ is Borel and thus the function ψ defined above is a Borel function. □

6.2. **(Not) Borel Embedding Height Three Partial Orders.** We will now explain how to reproduce the results of section 4 in the Borel setting. Namely, we will show that there is a height three, locally countable Borel partial order on 2^ω which does not have a Borel embedding into Turing reducibility. The proof of Theorem 4.5 actually requires little modification to adapt to the Borel setting. Recall that the proof was broken into two parts: the first showed that in $\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$, no partial order with certain properties can be embedded into the Turing degrees and the second showed that in ZF there is a height three, locally countable partial order of size continuum with those properties.

A close examination of the proof of the second part (the construction of the height three partial order) reveals that it actually yields a Borel partial order on 2^ω. For the first part, the only fact used in the proof that is not provable in ZF is that every uncountable subset of 2^ω contains a perfect set. Fortunately, there is a replacement for this fact in the Borel setting. Namely, every uncountable $\mathbf{\Sigma^1_1}$ subset of 2^ω contains a perfect set.

Theorem 6.5 (Perfect set theorem for analytic sets; [Kec95] Exercise 14.13)**.** *Every $\mathbf{\Sigma^1_1}$ definable subset of 2^ω is either countable or contains a perfect set.*

We will now give the proofs of the Borel versions of Lemmas 4.3 and 4.4 in greater detail.

Lemma 6.6. *Suppose $\leq_P$ is a height three Borel partial order on 2^ω with the following properties.*

1. *The first level of $\leq_P$ (i.e. the elements of 2^ω which have no $\leq_P$-predecessors) has size continuum.*
2. *Every countable subset of the first level of $\leq_P$ has an upper bound in the second level.*
3. *For every finite subset $S \subset 2^\omega$ which is contained in the first two levels of $\leq_P$ and every $x \in 2^\omega$ in the second level which is not contained in S, there is some $y \in 2^\omega$ in the third level which is above every element of S but not above x.*

Then there is no Borel embedding of $\leq_P$ into Turing reducibility.

Proof. Suppose for contradiction that f is a Borel embedding of $\leq_P$ into $\leq_T$. Just as in the proof of Lemma 4.3, we want to begin by showing that the image of the first level of $\leq_P$ under f contains a perfect set. To do so, we just use the Lusin-Novikov theorem to show that the first level of $\leq_P$ is Borel and then use the perfect set theorem for $\mathbf{\Sigma^1_1}$ sets to show that the image of the first level under f must contain a perfect set. Let's now explain slightly more carefully how this works.

Let P_0, P_1 and P_2 denote the first, second and third levels of $\leq_P$, respectively. By the first part of the Lusin-Novikov theorem, the set

$$P_1 \cup P_2 = \{x \mid \exists y \, (y <_P x)\}$$

is Borel (this uses the fact that $\leq_P$ is locally countable). Therefore $P_0 = 2^\omega \setminus (P_1 \cup P_2)$ is also Borel. And since f is Borel, $f(P_0)$ is $\mathbf{\Sigma^1_1}$ definable. Since P_0 is uncountable and f is injective, the perfect set theorem for $\mathbf{\Sigma^1_1}$ sets implies that $f(P_0)$ contains a perfect set.

The rest of the proof is identical to the proof of Lemma 4.3. □

Lemma 6.7. *There is a height three, locally countable Borel partial order on 2^ω with the properties listed in the statement of Lemma 6.6.*

Proof. The partial order described in the proof of Lemma 4.4 is actually Borel, but it is not a partial order on 2^ω. In some sense this is an artificial difficulty because everything that we are doing in the section still works for Borel partial orders on arbitrary Polish spaces; we only restricted to the case of Borel partial orders on 2^ω to keep things simple. In any case, it is easy to adapt the proof of Lemma 4.4 to give a Borel partial order on 2^ω, as we will now explain.

By picking a bijection $f\colon \omega \to \omega \times \omega$, we can think of elements of 2^ω as elements of $(2^\omega)^\omega$. If x is an element of 2^ω then we will use the phrase "the n^{th} column of x" to refer to the element of 2^ω defined by $\{m \in \omega \mid f(n,m) \in x\}$, which we will also write as x_n. Also, if $x \in 2^\omega$, let $1^\frown x$ denote the element of 2^ω obtained by appending 1 at the front of x (i.e. $1^\frown x = \{0\} \cup \{n > 0 \mid n - 1 \in x\}$) and similarly for $11^\frown x$. We will assume that the first two bits of the first column of each real are the first two bits of the real (i.e. that $f(0,0) = 0$ and $f(0,1) = 1$).

Now define three sets: P_0, P_1, and P_2. P_0 is the set of elements of 2^ω whose first column starts with 11, P_1 is the set of elements of 2^ω whose first column starts with 10 and P_2 is the set of elements of 2^ω whose first column starts with 0. Now define a relation $\leq_P$ on 2^ω by setting $x \leq_P y$ if and only if one of the following holds.

- $x = y$
- y is in P_1 and x is in P_0 and for some $n > 0$, $11^\frown y_n = x$
- y is in P_2 and x is in P_0 or P_1 and for some n, $1^\frown y_n = x$.
- y is in P_2 and x is in P_0 and for some z in P_1, $x \leq_P z \leq_P y$ by one of the two previous rules.

The explicit definition given above makes it clear that $\leq_P$ is Borel and it is easy to check that $\leq_P$ has the required properties (one key point is that the specific definition of P_0, P_1 and P_2 given above guarantees that every element of P_2 is above at least one element of P_1—recall that the first column of an element of P_2 begins with 0 and thus when we append a 1 to the beginning of that column we get an element of P_1). □

The Borel version of Theorem 4.5 now follows from the previous two lemmas.

Theorem 6.8. *There is a height three, locally countable Borel partial order on 2^ω which has no Borel embedding into Turing reducibility.*

7. Connection to Countable Borel Equivalence Relations

There is something a little odd about the results of the previous section. Namely, the two main theorems of the previous section concerned whether or not certain Borel partial orders on the reals have Borel embeddings into Turing reducibility. But Turing reducibility is not itself a partial order on the reals. Instead, it is a quasi order (essentially a partial order where some elements are allowed to be equivalent to each other, but literally just a transitive, reflexive binary relation). This suggests that it may be more natural to phrase our results in terms of the theory of locally countable Borel quasi orders. And since the theory of locally countable Borel quasi orders in some ways parallels the more well-studied theory of countable Borel equivalence relations, this also suggests that we compare our results to what is known in that theory.

We will start with a few formal definitions. A **quasi order** is simply a binary relation which is transitive and reflexive. If $(P, \leq_P)$ is a quasi order then the relation $x \sim_P y$ which holds whenever both $x \leq_P y$ and $y \leq_P x$ hold is an equivalence relation on P and $\leq_P$ is a partial order on the quotient of P by this equivalence relation (just as Turing reducibility is a partial order on the quotient of the reals by Turing equivalence). A quasi order $(P, \leq_P)$ is called **locally countable** if for each $x \in P$, the set of predecessors of x (i.e. the set $\{y \in P \mid y \leq_P x\}$) is countable. Note that this set includes the elements of P which are equivalent to x.

On the other hand, a **countable equivalence relation** is an equivalence relation whose equivalence classes are all countable. Note that if $(P, \leq_P)$ is a locally countable quasi order then the equivalence relation, $\sim_P$, associated to $\leq_P$ is a countable equivalence relation.

We will also define formally what it means for a locally countable Borel quasi order or a countable Borel equivalence relation to be **universal**.

Definition 7.1. *A locally countable Borel quasi order on 2^ω is called **universal** if every locally countable Borel quasi order on 2^ω has a Borel embedding into it (in the sense of Definition 6.1).*

Definition 7.2. *A countable Borel equivalence relation on 2^ω is called **universal** if every countable Borel equivalence relation on 2^ω has a Borel embedding into it.*

A major open question in the theory of countable Borel equivalence relations is whether Turing equivalence is universal. This question is interesting both for its own sake and because it contradicts Martin's conjecture, a major open question in computability theory; see [MSS16] for more about this.

Conjecture 7.3 (Kechris; [DK00]). *Turing equivalence is a universal countable Borel equivalence relation.*

Since every partial order is also a quasi order, Theorem 6.8 shows that Turing reducibility is *not* a universal locally countable Borel quasi order.

Theorem 7.4. *Turing reducibility is not a universal locally countable Borel quasi order.*

For other locally countable Borel quasi orders, universality as a locally countable quasi order and as a countable equivalence relation seem to go together. For example, the "hereditarily a column of" relation and arithmetic reducibility are both universal as locally countable Borel quasi orders and as countable Borel equivalence relations. Thus the above theorem seems to be evidence against Kechris' conjecture.

In fact, the results of this paper actually seem to provide more evidence against Kechris' conjecture than just the fact that Turing reducibility is not a universal locally countable Borel quasi order. Let us explain. Define the height of a quasi order to be the length of the longest strictly decreasing chain (i.e. the height of the partial order after quotienting by the associated equivalence relation). Note that a countable equivalence relation can be thought of as a locally countable quasi order of height one. Embedding an equivalence relation into Turing equivalence (as equivalence relations) is not quite the same as embedding that same equivalence relation into Turing reducibility (as quasi orders)—in the former case, distinct equivalence classes only need to be sent to distinct Turing degrees, but in the latter case they must be sent to *incomparable* degrees—but it is similar. Thus, one can view the statement that every locally countable Borel quasi order of height one has a Borel embedding into Turing reducibility as a mild strengthening of Kechris' conjecture. In this paper, we have shown that this fails if "height one" is replaced by "height three."

At this point some readers may object that, under this framing, the results of this paper may actually be seen as support for Kechris' conjecture. After all, Theorem 6.4 shows that every height two, locally countable Borel partial order has a Borel embedding into Turing reducibility. Doesn't this suggest that it may be possible to do the same for *quasi* orders of height *one*? In our opinion, however, this evidence is not very convincing. In a quasi order of height one, it is possible to have an infinitely long chain of distinct elements which are all related to each other, but this is impossible in a partial order of finite height. Thus we wish to end with the following conjecture.

Conjecture 7.5. *There is a height one, locally countable Borel quasi order on* 2^ω *which has no Borel embedding into Turing reducibility.*

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