MATH 54 SUMMER 2017, QUIZ 31

Suppose that f and g are continuous functions defined on the interval $[-\pi, \pi]$ and that for any positive integer n,

$$\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = 0 \qquad \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{n^2} \qquad \int_{-\pi}^{\pi} f(x)\,dx = 1$$

$$\int_{-\pi}^{\pi} g(x)\sin(nx)\,dx = 0 \qquad \int_{-\pi}^{\pi} g(x)\cos(nx)\,dx = \frac{1}{n^3} \qquad \int_{-\pi}^{\pi} g(x)\,dx = 2$$

Find the Fourier series of $3f(x)-5g(x)$. [Hint: for any positive integer n, $\int_{-\pi}^{\pi}\cos^2(nx)\,dx = \pi$ and $\int_{-\pi}^{\pi} 1\,dx = 2\pi$.]

Fourier series of $3f-5g$ is:

$$a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right)$$

where

$$a_0 = \frac{\langle 3f-5g, 1\rangle}{\langle 1, 1\rangle} \qquad a_n = \frac{\langle 3f-5g, \cos(nx)\rangle}{\langle \cos(nx), \cos(nx)\rangle}$$

$$b_n = \frac{\langle 3f-5g, \sin(nx)\rangle}{\langle \sin(nx), \sin(nx)\rangle}$$

$$a_0 = \frac{\langle 3f-5g, 1\rangle}{\langle 1, 1\rangle} = \frac{3\langle f, 1\rangle - 5\langle g, 1\rangle}{\langle 1, 1\rangle} = \frac{3\int_{-\pi}^{\pi} f(x)dx - 5\int_{-\pi}^{\pi} g(x)dx}{\int_{-\pi}^{\pi} 1^2 dx} = \frac{3-10}{2\pi} = -\frac{7}{2\pi}$$

$$a_n = \frac{\langle 3f-5g, \cos(nx)\rangle}{\langle \cos(nx), \cos(nx)\rangle} = \frac{3\langle f, \cos(nx)\rangle - 5\langle g, \cos(nx)\rangle}{\langle \cos(nx), \cos(nx)\rangle} = \frac{3\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx - 5\int_{-\pi}^{\pi} g(x)\cos(nx)\,dx}{\int_{-\pi}^{\pi}\cos^2(nx)\,dx}$$

$$= \frac{3/n^2 - 5/n^3}{\pi} = \frac{3n-5}{\pi n^3}$$

$$b_n = \frac{\langle 3f-5g, \sin(nx)\rangle}{\langle \sin(nx), \sin(nx)\rangle} = \frac{3\langle f, \sin(nx)\rangle - 5\langle g, \sin(nx)\rangle}{\langle \sin(nx), \sin(nx)\rangle} = \frac{3\int_{-\pi}^{\pi} f(x)\sin(nx)dx - 5\int_{-\pi}^{\pi} g(x)\sin(nx)dx}{\int_{-\pi}^{\pi}\sin^2(nx)dx}$$

$$= 0$$

So the Fourier series is $\boxed{-\frac{7}{2\pi} + \sum_{n=1}^{\infty}\left(\frac{3n-5}{\pi n^3}\right)\cos(nx)}$