MATH 54 SUMMER 2017, QUIZ 30

(a) Find all eigenvectors of $\frac{d^3}{dx^3} - 9\frac{d^2}{dx^2} + 27\frac{d}{dx}$ with eigenvalue 27. [Hint: the polynomial $t^3 - 9t^2 + 27t - 27$ can be factored as $(t-3)^3$.]

$f(x)$ is an eigenvector of $\frac{d^3}{dx^3} - 9\frac{d^2}{dx^2} + 27\frac{d}{dx}$ with eigenvalue 27 if and only if

$$\frac{d^3 f(x)}{dx^3} - 9\frac{d^2 f(x)}{dx^2} + 27\frac{df(x)}{dx} = 27f(x).$$

In other words, if and only if $f(x)$ is a solution to the ODE:

$$y''' - 9y'' + 27y' - 27y = 0$$

Auxiliary equation: $r^3 - 9r^2 + 27r - 27 = 0$

$(r-3)^3 = 0$ (by the hint)

Solutions: $\boxed{c_1 e^{3x} + c_2 x e^{3x} + c_3 x^2 e^{3x}}$

where c_1, c_2, c_3 are any scalars not all 0 (since eigenvectors are supposed to be nonzero).

(b) Use your answer to part (a) to find a nontrivial solution to the following PDE.

$$\frac{\partial f}{\partial t} = \frac{\partial^3 f}{\partial x^3} - 9\frac{\partial^2 f}{\partial x^2} + 27\frac{\partial f}{\partial x}$$

Let T be the linear transformation $\frac{\partial^3}{\partial x^3} - 9\frac{\partial^2}{\partial x^2} + 27\frac{\partial}{\partial x}$.

So the PDE is

$$\frac{\partial f}{\partial t} = T(f).$$

We saw in class that if $y(x)$ is an eigenvector of T with eigenvector λ then $e^{\lambda t} y(x)$ is a solution to the above PDE.

So by part (a), $\boxed{e^{27t}\left(c_1 e^{3x} + c_2 x e^{3x} + c_3 x^2 e^{3x}\right)}$

is a solution for any $c_1, c_2, c_3 \in \mathbb{R}$ (and it is a nontrivial solution when c_1, c_2, c_3 are not all 0).