MATH 54 SUMMER 2017, QUIZ 23

Find the projection of $\sin(x)$ on the subspace $\operatorname{span}\{1, x\}$ in the inner product space $C([0,\pi])$ with the inner product given below.

$$\langle f, g\rangle = \int_0^\pi f(x)g(x)\,dx$$

Warning: with the inner product above, $\{1, x\}$ is *not* an orthogonal set!

[Hint: $\int_0^\pi \sin(x)\,dx = 2$ and $\int_0^\pi x\sin(x)\,dx = \pi$.]

① Use Gram-Schmidt to find an orthogonal basis for $\operatorname{span}\{1, x\}$

$f_1 = 1$

$f_2 = x - \operatorname{proj}_{\operatorname{span}\{f_1\}}(x) = x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle} 1 = x - \frac{\pi^2/2}{\pi} 1 = x - \frac{\pi}{2}$

$$\boxed{\begin{aligned} \langle 1, 1\rangle &= \int_0^\pi 1\,dx = \pi \\ \langle x, 1\rangle &= \int_0^\pi x\,dx = \frac{x^2}{2}\Big|_0^\pi = \frac{\pi^2}{2} \end{aligned}}$$

orthogonal basis: $\{1, x - \frac{\pi}{2}\}$

② Find the projection of $\sin(x)$ using this orthogonal basis

$$\begin{aligned} \operatorname{proj}_{\operatorname{span}\{1,x\}}(\sin(x)) &= \operatorname{proj}_{f_1}(\sin(x)) + \operatorname{proj}_{f_2}(\sin(x)) \\ &= \frac{\langle \sin(x), 1\rangle}{\langle 1, 1\rangle} 1 + \frac{\langle \sin(x), x - \frac{\pi}{2}\rangle}{\langle x - \frac{\pi}{2}, x - \frac{\pi}{2}\rangle}\left(x - \frac{\pi}{2}\right) \\ &= \frac{2}{\pi} 1 + \frac{0}{\langle x - \frac{\pi}{2}, x - \frac{\pi}{2}\rangle}\left(x - \frac{\pi}{2}\right) \\ &= \boxed{\frac{2}{\pi}} \end{aligned}$$

$$\boxed{\begin{aligned} \langle \sin(x), 1\rangle &= \int_0^\pi \sin(x)\,dx = 2 \\ \langle \sin(x), x - \tfrac{\pi}{2}\rangle &= \langle \sin(x), x\rangle - \tfrac{\pi}{2}\langle \sin(x), 1\rangle \\ &= \int_0^\pi x\sin(x)\,dx - \tfrac{\pi}{2}\cdot 2 \\ &= \pi - \tfrac{\pi}{2}\cdot 2 \\ &= 0 \end{aligned}}$$

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