MATH 54 SUMMER 2017, QUIZ 22

Find all least squares solutions to

$$\begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}.$$

Let $A = \begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$.

Then $A^T = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+1+4 & -3+2 \\ -3+2 & 9+1 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ -1 & 10 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5-2+2 \\ 6+1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

<u>Normal equations:</u> $A^TA\hat{x} = A^T\vec{b}$

$$\left[\begin{array}{cc|c} 6 & -1 & 5 \\ -1 & 10 & 7 \end{array}\right] \xrightarrow{\text{Swap } R1 \,\&\, R2} \left[\begin{array}{cc|c} -1 & 10 & 7 \\ 6 & -1 & 5 \end{array}\right] \xrightarrow{R2 = R2 + 6R1} \begin{bmatrix} -1 & 10 & 7 \\ 0 & 59 & 47 \end{bmatrix}$$

$$\begin{aligned} -x_1 + 10x_2 &= 7 \\ 59x_2 &= 47 \end{aligned} \Rightarrow \begin{aligned} x_1 &= -7 + 10x_2 = -7 + \frac{470}{59} = \frac{470-417}{59} = \frac{57}{59} \\ x_2 &= 47/59 \end{aligned}$$

So the unique least-squares solution is $\boxed{\begin{bmatrix} 57/59 \\ 47/59 \end{bmatrix}}$

<u>Alternative Method:</u>

$$\hat{x} = (A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} 6 & -1 \\ -1 & 10 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$= \frac{1}{60-1}\begin{bmatrix} 10 & 1 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} = \frac{1}{59}\begin{bmatrix} 57 \\ 47 \end{bmatrix}$$

A^TA is invertible because its determinant, is $6 \cdot 10 - (-1)(-1) = 59 \neq 0$

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