Math 54 Midterm, Summer 2017

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Name: ____

PLEDGE: I promise I will not cheat on this exam in any way.

Sign Here: ____

INSTRUCTIONS: Answer each question in the space provided. If you run out of room, use the blank pages at the end. Good luck and, as it says on the cover of the *The Hitchhiker's Guide to the Galaxy*, Don't Panic.

Question	Points	Score
1	5	
2	4	
3	10	
4	4	
5	5	
6	6	
7	4	
8	4	
9	8	
Total:	50	

Don't turn over this page until you are told to do so.

1. (5 points) Are the following vectors linearly independent? If so, explain why. If not, find a nontrivial linear combination that is equal to zero.

$$\begin{bmatrix}1\\0\\2\\0\end{bmatrix}, \quad \begin{bmatrix}3\\-3\\2\\1\end{bmatrix}, \quad \begin{bmatrix}-1\\3\\2\\-1\end{bmatrix}$$

Solution:

$$\begin{bmatrix}1 & 3 & -1\\0 & -3 & 3\\2 & 2 & 2\\0 & 1 & -1\end{bmatrix} \xrightarrow{R3=R3-2R1} \begin{bmatrix}1 & 3 & -1\\0 & -3 & 3\\0 & -4 & 4\\0 & 1 & -1\end{bmatrix} \xrightarrow{R2=-\frac{1}{3}R2} \begin{bmatrix}1 & 3 & -1\\0 & 1 & -1\\0 & -4 & 4\\0 & 1 & -1\end{bmatrix}$$

$$\xrightarrow{R3=R3+4R2} \begin{bmatrix}1 & 3 & -1\\0 & 1 & -1\\0 & 0 & 0\\0 & 1 & -1\end{bmatrix} \xrightarrow{R4=R4-R2} \begin{bmatrix}1 & 3 & -1\\0 & 1 & -1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

$$\xrightarrow{R1=R1-3R2} \begin{bmatrix}1 & 0 & 2\\0 & 1 & -1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

Since the above matrix only has two pivots, the vectors are $\boxed{\text{not linearly independent.}}$

To find a nontrivial linear combination equal to zero, we can solve the homogeneous equation corresponding to the above matrix. The solution is

$$\begin{aligned} x_3 &\text{ is free} \\ x_2 - x_3 = 0 &\implies x_2 = x_3 \\ x_1 + 2x_3 = 0 &\implies x_1 = -2x_3 \end{aligned}$$

Arbitrarily choosing $x_3 = 1$ gives us $\boxed{x_1 = -2,\ x_2 = 1,\ x_3 = 1.}$ In other words:

$$-2\begin{bmatrix}1\\0\\2\\0\end{bmatrix} + \begin{bmatrix}3\\-3\\2\\1\end{bmatrix} + \begin{bmatrix}-1\\3\\2\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

2. (4 points) Find a basis for the eigenspace of the eigenvalue 3 of the matrix A shown below.

$$A = \begin{bmatrix} 6 & -6 & 9 \\ 2 & -1 & 6 \\ -1 & 2 & 0 \end{bmatrix}$$

Solution: $E_3 = \text{Null}(A - 3I)$.

$$A - 3I = \begin{bmatrix} 3 & -6 & 9 \\ 2 & -4 & 6 \\ -1 & 2 & -3 \end{bmatrix} \xrightarrow{R1=\frac{1}{3}R1} \begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ -1 & 2 & -3 \end{bmatrix}$$

$$\xrightarrow{R2=R2-2R1} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ -1 & 2 & -3 \end{bmatrix} \xrightarrow{R3=R3+R1} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solving the homogeneous equation of the above matrix gives us

$$x_3 \text{ is free}$$
$$x_2 \text{ is free}$$
$$x_1 - 2x_2 + 3x_3 = 0 \implies x_1 = 2x_2 - 3x_3$$

Writing this solution in parametric form gives us

$$x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Therefore a basis for E_3 is

$$\boxed{\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

Let's check that both of these are eigenvectors of A.

$$A \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

$$A \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ 0 \\ 3 \end{bmatrix}$$

3. Let $\mathbb{P}_2 = \{a_2x^2 + a_1x + a_0 \mid a_0, a_1, a_2 \in \mathbb{R}\}$ denote the vector space of all polynomials of degree at most two. Let $T\colon \mathbb{P}_2 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(p) = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \end{bmatrix}.$$

Let $\mathcal{B} = \{x-1, x^2-1, x^2+x\}$ and let $\mathcal{E}$ be the standard basis for $\mathbb{R}^3$.

(a) (4 points) Find the matrix of T relative to the bases $\mathcal{B}$ and $\mathcal{E}$—i.e. find ${}_{\mathcal{E}}[T]_{\mathcal{B}}$.

Solution: First evaluate T on the vectors in $\mathcal{B}$:

$$T(x-1) = \begin{bmatrix} 0-1 \\ 1-1 \\ 2-1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$T(x^2-1) = \begin{bmatrix} 0-1 \\ 1-1 \\ 4-1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

$$T(x^2+x) = \begin{bmatrix} 0+0 \\ 1+1 \\ 4+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$$

Then we write each of these as coordinate vectors in $\mathcal{E}$ to get the columns of ${}_{\mathcal{E}}[T]_{\mathcal{B}}$. Since $\mathcal{E}$ is the standard basis of $\mathbb{R}^3$, this gives us

$$\boxed{{}_{\mathcal{E}}[T]_{\mathcal{B}} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & 3 & 6 \end{bmatrix}}$$

(b) (4 points) Find the inverse of the matrix you found in part (a).

Solution:

$$\left[\begin{array}{ccc|ccc} -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R3 = R3 + R1} \left[\begin{array}{ccc|ccc} -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 6 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{Swap } R2 \text{ and } R3} \left[\begin{array}{ccc|ccc} -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 6 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R3 = \frac{1}{2}R3} \left[\begin{array}{ccc|ccc} -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 6 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1/2 & 0 \end{array}\right]$$

$$\xrightarrow{R2=R2-6R3} \left[\begin{array}{ccc|ccc} -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & 0 & 1/2 & 0 \end{array}\right]$$

$$\xrightarrow{R2=\frac{1}{2}R2} \left[\begin{array}{ccc|ccc} -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & -3/2 & 1/2 \\ 0 & 0 & 1 & 0 & 1/2 & 0 \end{array}\right]$$

$$\xrightarrow{R1=R1+R2} \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 3/2 & -3/2 & 1/2 \\ 0 & 1 & 0 & 1/2 & -3/2 & 1/2 \\ 0 & 0 & 1 & 0 & 1/2 & 0 \end{array}\right]$$

$$\xrightarrow{R1=-R1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/2 & 3/2 & -1/2 \\ 0 & 1 & 0 & 1/2 & -3/2 & 1/2 \\ 0 & 0 & 1 & 0 & 1/2 & 0 \end{array}\right]$$

So the inverse is:

$$\boxed{\begin{bmatrix} -3/2 & 3/2 & -1/2 \\ 1/2 & -3/2 & 1/2 \\ 0 & 1/2 & 0 \end{bmatrix}}$$

Let's check this:

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} -3/2 & 3/2 & -1/2 \\ 1/2 & -3/2 & 1/2 \\ 0 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) (2 points) Find a degree two polynomial p such that $p(0) = -2$, $p(1) = 2$, and $p(2) = 10$.

Solution: Finding such a polynomial is equivalent to finding

$$T^{-1}\left(\begin{bmatrix} -2 \\ 2 \\ 10 \end{bmatrix}\right).$$

By part (b), we know ${}_{\mathcal{E}}[T]_{\mathcal{B}}^{-1}$. So we know that

$$\left[T^{-1}\left(\begin{bmatrix} -2 \\ 2 \\ 10 \end{bmatrix}\right)\right]_{\mathcal{B}} = \begin{bmatrix} -3/2 & 3/2 & -1/2 \\ 1/2 & -3/2 & 1/2 \\ 0 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ 10 \end{bmatrix} = \begin{bmatrix} 3+3-5 \\ -1-3+5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

To find the solution we need to find the polynomial whose coordinate vector in $\mathcal{B}$ is the vector above. This polynomial is

$$(x-1) + (x^2-1) + (x^2+x) = \boxed{2x^2+2x-2}$$

Let's check this:

$$\begin{aligned} 2(0)^2 + 2(0) - 2 &= -2 \\ 2(1)^2 + 2(1) - 2 &= 2 \\ 2(2)^2 + 2(2) - 2 &= 10 \end{aligned}$$

4. (4 points) Find a diagonal matrix similar to A^{2017} where

$$A = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}.$$

Solution: To solve this problem we will first find a diagonal matrix similar to A. To do this we need to find the eigenvalues of A.

$$\begin{aligned} \det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 4 \\ 4 & 1-\lambda \end{bmatrix}\right) &= (1-\lambda)^2 - 16 = \lambda^2 - 2\lambda - 15 \\ &= (\lambda-5)(\lambda+3) \end{aligned}$$

So the eigenvalues of A are 5 and -3. Since A is 2×2 and has 2 distinct eigenvalues, it is diagonalizable and is similar to $D = \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}$. In other words, for some invertible P, $A = PDP^{-1}$. Therefore $A^{2017} = PD^{2017}P^{-1}$. So A^{2017} is similar to $\boxed{\begin{bmatrix} 5^{2017} & 0 \\ 0 & (-3)^{2017} \end{bmatrix}}$

5. (5 points) What is the determinant of the following matrix? Briefly justify your answer.

$$\begin{bmatrix} 1 & 7 & 8 & 1 & 2 & 3 \\ 2 & -9 & 81 & 2 & 7 & 0 \\ 3 & 4 & 7 & 3 & 7 & -1 \\ 4 & 1 & 1 & 4 & 1 & 1 \\ 5 & 7 & -3 & 5 & 13 & 788 \\ 6 & -1 & -2 & 6 & -4 & -5 \end{bmatrix}$$

Solution: Since the first and fourth columns are equal, the columns of the above matrix are linearly dependent. Therefore the matrix is not invertible and so its determinant is $\boxed{0}$.

6. (a) (2 points) If A is a 3×5 matrix, what is the smallest possible value of $\dim(\operatorname{Null} A)$? You do not need to explain your reasoning.

Solution: A has three rows, so in REF it can have at most three pivots. Therefore the rank is at most three. Since $\operatorname{rank}(A) + \dim(\operatorname{Null} A) = 5$, this means $\dim(\operatorname{Null} A)$ is at least $\boxed{2}$.

(b) (1 point) Give an example of a 3×5 matrix A where $\dim(\operatorname{Null} A)$ is as small as possible.

Solution: We need a matrix with rank 3. There are many possible examples, one is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(c) (2 points) If A is a 3×5 matrix, what is the largest possible value of $\dim(\operatorname{Null} A)$? You do not need to explain your reasoning.

Solution: Since $\operatorname{Null} A$ is a subspace of $\mathbb{R}^5$, its dimension is at most $\boxed{5}$.

(d) (1 point) Give an example of a 3×5 matrix A where $\dim(\operatorname{Null} A)$ is as large as possible.

Solution: We need a matrix whose null space is all of $\mathbb{R}^5$. This means that the matrix multiplied by any vector in $\mathbb{R}^5$ must give zero. The only matrix satisfying this condition is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7. (4 points) Suppose $T\colon \mathbb{P}_2 \to \mathbb{P}_1$ is a linear transformation such that $T(5x+2) = 3x$ and $\ker(T) = \operatorname{span}\{x^2+2x\}$. Find two distinct polynomials p and q in $\mathbb{P}_2$ such that $T(p) = T(q) = 3x$.

Solution: We know one solution to $T(p) = 3x$, we know the kernel of T and we need to find two solutions to $T(p) = 3x$. All solutions to $T(p) = 3x$ can be found by adding the particular solution we know to any polynomial in the kernel. So for instance $(5x+2)+(x^2+2x) = \boxed{x^2+7x+2}$ and $(5x+2)-7(x^2+2x) = \boxed{-7x^2-9x+2}$ are two possible answers. Of course, $\boxed{5x+2}$ is also a valid answer (this reflects the fact that 0 is in the kernel of T).

8. (4 points) Suppose $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathcal{C}$ are bases for $\mathbb{R}^2$ and $\mathcal{D}$ is the basis for $\mathbb{R}^2$ shown below. Find $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$\mathcal{D} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\} \quad \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \quad \underset{\mathcal{D}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

Solution: Recall that if $\mathcal{E}$ is the standard basis of $\mathbb{R}^2$, the columns of $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$ are the vectors in $\mathcal{B}$. We will find $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$ as follows:

$$\begin{aligned}
\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{E}\leftarrow\mathcal{D}}{P}\,\underset{\mathcal{D}\leftarrow\mathcal{C}}{P}\,\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} &= \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \underset{\mathcal{D}\leftarrow\mathcal{C}}{P}\,\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \\
&= \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 10 \\ 2 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 7 & 22 \\ 2 & 4 \end{bmatrix}
\end{aligned}$$

Therefore

$$\boxed{\mathbf{v}_1 = \begin{bmatrix} 7 \\ 2 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} 22 \\ 4 \end{bmatrix}}$$

9. Mark each of the following true or false. If true, briefly explain why. If false, give a counterexample.

(a) (2 points) For any $n \times m$ matrix A and any $m \times p$ matrix B, if $\operatorname{Col} AB = \mathbb{R}^n$ then $\operatorname{Col} A = \mathbb{R}^n$.

Solution: True. We did a very similar problem in class at some point. The main idea is to think of A and B as linear transformations and to realize that the range of AB cannot be larger than the range of A.

More formally: since $\operatorname{Col} AB = \mathbb{R}^n$, the linear transformation $\mathbf{x} \mapsto AB\mathbf{x}$ is onto. We need to show that the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto. Let $\mathbf{y}$ be any vector in $\mathbb{R}^n$. Since the linear transformation given by AB is onto, there is some $\mathbf{x}$ such that $AB\mathbf{x} = \mathbf{y}$. So $A(B\mathbf{x}) = \mathbf{y}$ and therefore $\mathbf{y}$ is in the range of the linear transformation given by A. This means that the linear transformation given by A is onto and, equivalently, that $\operatorname{Col} A = \mathbb{R}^n$.

(b) (2 points) For any linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and any $n \times n$ matrix A, $A\mathbf{u}$ and $A\mathbf{v}$ are linearly independent.

Solution: False. There are many counterexamples to this (any non-invertible matrix works). One is

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(c) (2 points) For any basis $\{\mathbf{u}, \mathbf{v}\}$ for $\mathbb{R}^2$ and any $n \times 2$ matrices A and B, if $A\mathbf{u} = B\mathbf{u}$ and $A\mathbf{v} = B\mathbf{v}$ then $A = B$.

Solution: True. The main idea here is that if you know what a linear transformation does to a basis, you know what it does to everything.

More formally: since $\mathbf{u}$ and $\mathbf{v}$ span all of $\mathbb{R}^2$, both $\mathbf{e}_1$ and $\mathbf{e}_2$ can be written as linear combinations of $\mathbf{u}$ and $\mathbf{v}$. So for some real numbers a and b, $\mathbf{e}_1 = a\mathbf{u} + b\mathbf{v}$. Therefore

$$A\mathbf{e}_1 = A(a\mathbf{u} + b\mathbf{v}) = aA\mathbf{u} + bA\mathbf{v} = aB\mathbf{u} + bB\mathbf{v} = B(a\mathbf{u} + b\mathbf{v}) = B\mathbf{e}_1.$$

By the same reasoning, $A\mathbf{e}_2 = B\mathbf{e}_2$. Since a matrix times $\mathbf{e}_1$ gives the first column of the matrix and a matrix times $\mathbf{e}_2$ gives the second column of the matrix, this means A and B have the same first and second columns and so they are equal.

(d) (2 points) For any 2×2 matrices A and B, if A and B are invertible then $AB = BA$.

Solution: False. There are many possible counterexamples. One is

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

One way to think about this particular counterexample is that you get a different result if you first rotate by $\pi/2$ radians counterclockwise and then expand in the horizontal direction by two than if you first expand in the horizontal direction by two and then rotate by $\pi/2$ radians counterclockwise.