Finding Solutions to the Heat Equation

Def The heat equation is the following PDE:

$$\boxed{\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2} + P(x,t) \qquad \text{where } \beta \text{ is some positive constant}}$$

We will often also impose the following conditions:

For all $t \geq 0$:	$u(0,t) = 0$ $u(L,t) = 0$	(Boundary Values)
And for all x:	$u(x,0) = f(x)$	(Initial Value)

Think of L as the length of some wire and $f(x)$ as describing the initial temperature along the wire.

How will we solve this? We will use the same strategy that we used for solving ODEs.

Strategy

1. Find a basis for the set of solutions to the homogeneous heat equation (i.e. $\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$). Let's call the functions in this basis $u_1, u_2, \ldots$

(1.5) Get rid of any solutions from (1) that don't satisfy the boundary values.

2. Find a solution to the nonhomogeneous heat equation (i.e. $\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2} + P(x,t)$) which also satisfies the boundary values. Let's call this solution u_p.

3. Write the general solution to the nonhomogeneous heat equation:

$$u(x,t) = u_p + c_1 u_1 + c_2 u_2 + \ldots \qquad \text{where } c_1, c_2, \ldots \text{ are any scalars}$$

4. Find values of $c_1, c_2, \ldots$ such that:

$$f(x) = u_p(x,0) + c_1 u_1(x,0) + c_2 u_2(x,0) + \ldots \qquad \text{for all } 0 \leq x \leq L$$

Actually, depending on what $P(x,t)$ is, step (2) can be very hard and so we will not learn how to do it in this class. Today we will focus on steps (1) and (1.5)

Step ①:

Before we begin this step, let's recall how we solved systems of linear ODEs. In that case, the problem we wanted to solve was:

$$\frac{d}{dt}\vec{y}(t) = A\vec{y}(t).$$

Here, $\vec{y}(t)$ can be thought of as a function from $\mathbb{R} \to \mathbb{R}^n$. In other words, for each time t, $\vec{y}$ gives us a vector in $\mathbb{R}^n$. And A can be thought of as a linear transformation $\mathbb{R}^n \to \mathbb{R}^n$. We found solutions to this problem by finding eigenvectors of A. In particular, if $\vec{v}$ is an eigenvector of A with eigenvalue λ then

$$\vec{y}(t) = e^{\lambda t}\vec{v}$$

is a solution.

We will use the same method to find solutions to the homogeneous heat equation. In particular, we will think of $u(x,t)$ as a function from $\mathbb{R} \to C^\infty([0,L])$. In other words, for each time t, u gives us a vector in $C^\infty([0,L])$ (the vector space of infinitely differentiable functions $[0,L] \to \mathbb{R}$). We will also think of $\beta\frac{\partial^2}{\partial x^2}$ as a linear transformation from $C^\infty([0,L]) \to C^\infty([0,L])$.

Let's write these two problems side by side.

System of linear ODEs	Homogeneous Heat Equation
$\vec{y}: \mathbb{R} \to \mathbb{R}^n$	$u: \mathbb{R} \to C^\infty([0,L])$
$A: \mathbb{R}^n \to \mathbb{R}^n$ a linear transformation	$\beta\frac{\partial^2}{\partial x^2}: C^\infty([0,L]) \longrightarrow C^\infty([0,L])$ ↳ this is a linear transformation
Solve $\frac{d}{dt}\vec{y}(t) = A\vec{y}(t)$	Solve $\frac{\partial}{\partial t}u(x,t) = \beta\frac{\partial^2}{\partial x^2}u(x,t)$

This suggests we can find solutions to the heat equation by finding eigenvectors of $\beta\frac{\partial^2}{\partial x^2}$. And indeed we can!

Thm If $z(x)$ is an eigenvector of $\beta\frac{\partial^2}{\partial x^2}$ with eigenvalue λ then $e^{\lambda t}z(x)$ is a solution to the homogeneous heat equation.

pf Since $z(x)$ is an eigenvector of $\beta\frac{\partial^2}{\partial x^2}$ with eigenvalue λ, we have

$$\beta\frac{\partial^2}{\partial x^2}\left(e^{\lambda t}z(x)\right) = e^{\lambda t}\beta\frac{\partial^2}{\partial x^2}\left(z(x)\right) \qquad \text{(since } e^{\lambda t} \text{ does not depend on } x\text{)}$$

$$= e^{\lambda t}(\lambda z(x)) \qquad \text{(by definition of eigenvector)}$$

$$= \lambda e^{\lambda t}z(x).$$

In addition,

$$\frac{\partial}{\partial t}\left(e^{\lambda t} z(x)\right) = z(x)\frac{\partial}{\partial t}\left(e^{\lambda t}\right) \qquad \text{(since } z(x) \text{ does not depend on } t\text{)}$$

$$= z(x)\left(\lambda e^{\lambda t}\right)$$

$$= \lambda e^{\lambda t} z(x).$$

Therefore

$$\frac{\partial}{\partial t}\left(e^{\lambda t} z(x)\right) = \beta \frac{\partial^2}{\partial x^2}\left(e^{\lambda t} z(x)\right)$$

so $e^{\lambda t} z(x)$ is a solution. $\square$

End of proof symbol

So let's find the eigenvectors of $\beta \frac{\partial^2}{\partial x^2}$!

Case 1: $z(x)$ is an eigenvector of $\beta\frac{\partial^2}{\partial x^2}$ with eigenvalue $\lambda > 0$.

By definition this means that

$$\beta \frac{\partial^2}{\partial x^2} z(x) = \lambda z(x).$$

In other words

$$\beta z'' = \lambda z$$

Or equivalently

$$\beta z'' - \lambda z = 0.$$

This is just a 2^{nd} order linear ODE, so we know how to solve it!

Auxiliary equation: $\beta r^2 - \lambda = 0$

roots: $\pm\sqrt{\frac{\lambda}{\beta}}$

Solutions: $c_1 e^{\sqrt{\lambda/\beta}\, x} + c_2 e^{-\sqrt{\lambda/\beta}\, x}$ for any $c_1, c_2 \in \mathbb{R}$

Since we want an eigenvector, which are supposed to be nonzero, we should also require that at least one of c_1, c_2 is nonzero.

Case 2: Eigenvalue $\lambda = 0$.

$$\beta z'' = 0$$

Auxiliary equation: $\beta r^2 = 0$

roots: 0 ← double root

Solutions: $c_1 e^{0x} + c_2 x e^{0x} = c_1 + c_2 x$

Case 3: Eigenvalue $\lambda < 0$.

$$\beta z'' = \lambda z$$

Auxiliary equation: $\beta r^2 - \lambda = 0$

roots: $r = \pm\sqrt{\lambda/\beta} = \pm i\sqrt{-\lambda/\beta}$

Since $\beta > 0$ and $\lambda < 0$, $\lambda/\beta < 0$ and so the roots are complex. So now the solutions are;

$$c_1 \cos\left(\sqrt{-\lambda/\beta}\, x\right) + c_2 \sin\left(\sqrt{-\lambda/\beta}\, x\right).$$

Now that we've found all the eigenvectors of $\beta \frac{\partial^2}{\partial x^2}$, we can write down solutions to the homogeneous heat equation. There are many of them and here they are:

① $e^{\lambda t}\left(c_1 e^{\sqrt{\lambda/\beta}\, x} + c_2 e^{-\sqrt{\lambda/\beta}\, x}\right)$ for any $\lambda > 0$

② $c_1 + c_2 x$

③ $e^{\lambda t}\left(c_1 \cos\left(\sqrt{-\lambda/\beta}\, x\right) + c_2 \sin\left(\sqrt{-\lambda/\beta}\, x\right)\right)$ for any $\lambda < 0$

We will now use the boundary values to get rid of some of these solutions.

Step (1.5): Eliminate solutions that don't satisfy the boundary values

Case 1: Suppose $\lambda > 0$. In order for $e^{\lambda t}\left(c_1 e^{\sqrt{\lambda/\beta}\, x} + c_2 e^{-\sqrt{\lambda/\beta}\, x}\right)$ to satisfy the boundary values, we must have:

$$\left.\begin{aligned} e^{\lambda t}\left(c_1 e^{\sqrt{\lambda/\beta}\cdot 0} + c_2 e^{-\sqrt{\lambda/\beta}\cdot 0}\right) &= 0 \\ e^{\lambda t}\left(c_1 e^{\sqrt{\lambda/\beta}\, L} + c_2 e^{-\sqrt{\lambda/\beta}\, L}\right) &= 0 \end{aligned}\right\} \text{ for all } t \geq 0$$

Since $e^{\lambda t}$ is never 0, this is equivalent to:

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 e^{\sqrt{\lambda/\beta}\, L} + c_2 e^{-\sqrt{\lambda/\beta}\, L} &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= -c_2 \\ -c_2 e^{\sqrt{\lambda/\beta}\, L} + c_2 e^{-\sqrt{\lambda/\beta}\, L} &= 0 \end{aligned}$$

$$\Rightarrow \quad c_2\left(e^{-\sqrt{\lambda/\beta}\, L} - e^{\sqrt{\lambda/\beta}\, L}\right) = 0$$

So either c_2 (and thus also c_1) is 0 or $e^{-\sqrt{\lambda/\beta}\, L} - e^{\sqrt{\lambda/\beta}\, L}$ is 0.

But if $e^{-\sqrt{\lambda/\beta}\,L} - e^{\sqrt{\lambda/\beta}\,L} = 0$ then

$$e^{-\sqrt{\lambda/\beta}\,L} = e^{\sqrt{\lambda/\beta}\,L}$$

$$\Rightarrow \quad -\sqrt{\lambda/\beta}\,L = \sqrt{\lambda/\beta}\,L$$

which is impossible since $\lambda \neq 0$ and $L \neq 0$. Therefore $c_1 = c_2 = 0$, so in this case we are just left with the trivial solution.

Case 2: If $c_1 + c_2 x$ satisfies the boundary conditions then

$$c_1 + c_2 \cdot 0 = 0$$
$$c_1 + c_2 \cdot L = 0$$

And therefore $c_1 = 0$ so $c_2 \cdot L = 0$, which means $c_2 = 0$. So in this case as well we are just left with the trivial solution.

Case 3: Suppose $\lambda < 0$. In order for $e^{\lambda t}\left(c_1 \cos\left(\sqrt{-\lambda/\beta}\,x\right) + c_2 \sin\left(\sqrt{-\lambda/\beta}\,x\right)\right)$ to satisfy the boundary values, we must have

$$\left.\begin{aligned} e^{\lambda t}\left(c_1 \cos\left(\sqrt{-\lambda/\beta}\cdot 0\right) + c_2 \sin\left(\sqrt{-\lambda/\beta}\cdot 0\right)\right) &= 0 \\ e^{\lambda t}\left(c_1 \cos\left(\sqrt{-\lambda/\beta}\cdot L\right) + c_2 \sin\left(\sqrt{-\lambda/\beta}\cdot L\right)\right) &= 0 \end{aligned}\right\} \text{ for all } t \geq 0$$

Since $e^{\lambda t}$ is never 0 and since $\sin(0) = 0$ and $\cos(0) = 1$, this implies

$$c_1 = 0$$
$$c_1 \cos\left(\sqrt{-\lambda/\beta}\,L\right) + c_2 \sin\left(\sqrt{-\lambda/\beta}\,L\right) = 0$$

and therefore

$$c_1 = 0$$
$$c_2 \sin\left(\sqrt{-\lambda/\beta}\,L\right) = 0.$$

So either c_1 and c_2 are both 0 or $\sin\left(\sqrt{-\lambda/\beta}\,L\right) = 0$. Since sin is only 0 at integer multiples of π, in order to get a nontrivial solution (i.e. a solution where c_1 and c_2 are not both 0) we must have

$$\sqrt{-\lambda/\beta}\,L = n\pi \quad \text{for some integer } n$$

or equivalently,

$$\lambda = -\beta\left(\frac{n\pi}{L}\right)^2 \qquad \text{for some integer } n.$$

So from all the solutions we started with, we are left with just the following:

$$C e^{\lambda t} \sin\left(\sqrt{-\lambda/\beta}\, x\right) \qquad \text{where } \lambda = -\beta\left(\frac{n\pi}{L}\right)^2 \text{ for some integer } n$$

and C is any nonzero real number

We can rewrite this as:

$$C e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L} x\right) \qquad \text{for any integer } n.$$

Also notice that n and $-n$ give the same eigenvalue. So the general solution to the homogeneous heat equation with the given boundary values is:

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L} x\right)$$

where the C_n are real numbers.

This formula may look long and scary, but let's recall where it came from:

$$C_n \, e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \, \sin\left(\frac{n\pi}{L} x\right)$$

Arbitrary constant because we can always take linear combinations of solutions

eigenvalue

eigenvector of $\beta \frac{\partial^2}{\partial x^2}$