Review

For each item below, explain why it is true or provide a counterexample to show it is false.

1. It is not possible to find n vectors in $\mathbb{R}^n$ that are linearly dependent.
2. Every list of n vectors in $\mathbb{R}^n$ spans all of $\mathbb{R}^n$.
3. If a list of n vectors in $\mathbb{R}^n$ is linearly independent then it spans all of $\mathbb{R}^n$.
4. If a list of n vectors in $\mathbb{R}^n$ spans all of $\mathbb{R}^n$ then it is linearly independent.

Linear Transformations

1. For each matrix below, make a drawing for the function from $\mathbb{R}^2 \to \mathbb{R}^2$ that it defines.

   (a) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$

   (b) $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. Is the function $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ defined by

   $$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} xy \\ y \\ x \end{bmatrix}$$

   a linear transformation?

3. If $T\colon \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation and $\mathbf{x}_1, \ldots, \mathbf{x}_p$ are linearly dependent vectors in $\mathbb{R}^m$ then are $T(\mathbf{x}_1), \ldots, T(\mathbf{x}_p)$ linearly dependent?

4. If $T\colon \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation and $\mathbf{x}_1, \ldots, \mathbf{x}_p$ are linearly independent vectors in $\mathbb{R}^m$ then are $T(\mathbf{x}_1), \ldots, T(\mathbf{x}_p)$ linearly independent?

5. If $T\colon \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation and $\mathbf{x}_1, \ldots, \mathbf{x}_p$ are vectors in $\mathbb{R}^m$ whose span is all of $\mathbb{R}^m$ then do $T(\mathbf{x}_1), \ldots, T(\mathbf{x}_p)$ span all of $\mathbb{R}^n$?

6. Suppose $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such that

   $$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

   What is $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$?

7. Write the standard matrix for each of the following linear transformations from $\mathbb{R}^2 \to \mathbb{R}^2$.

   (a) Reflection across the line $x_2 = x_1$.

   (b) Rotation by 90° followed by expansion by 3 in the horizontal direction.

   (c) Everything is sent to $\mathbf{0}$.

8. Is the linear transformation defined by the following matrix one-to-one? Onto?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

9. Is the linear transformation defined by the following matrix one-to-one? Onto?

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$