More Fourier Series

Start working on the first two questions below once you finish the quiz.

1. Find the Fourier series of the function $f\colon [-\pi,\pi] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} -1 & \text{when } x \le 0 \\ 1 & \text{when } x > 0 \end{cases}$$

2. What happens when you try evaluating the Fourier series from the previous question at 0? How about at π? Why do you think this is happening?

3. Suppose the Fourier series of an infinitely differentiable function f is

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \cos(nx)$$

(so the coefficients of the sine terms are zero).

   (a) What is $f(0)$?

   (b) What is $f(\pi)$?

   (c) What are the Fourier coefficients of $f(x) + \cos(x)$?

   (d) What are the fourier coefficients of $f(x)\cos(x)$?

Fourier Series and the Heat Equation

1. Find the Fourier cosine series of $\cos(2x) - 1$ on the interval $[0,\pi]$.

2. Find the Fourier sine series of $\cos(2x) - 1$ on the interval $[0,\pi]$.

3. Find a solution to the following PDE. [Hint: your solution to the previous question will be helpful here.]

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \\ &u(0,t) = u(\pi,t) = 0 \\ &u(x,0) = \cos(2x) - 1 \end{aligned}$$

4. Find a solution to the heat equation with initial condition $u(x,0) = 3f(x)$ where $f\colon [0,L] \to \mathbb{R}$ is a continuous function with piecewise continuous derivative (i.e. f is "nice enough") such that $f(0) = f(L) = 0$ and

$$\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx = \frac{1}{n^2}.$$

Definitions and Theorems

Definitions:

- Periodic function
- Fourier sine series, Fourier cosine series

Theorems:

- If $f\colon [0, L] \to \mathbb{R}$ is a "nice enough" function then the solution to the heat equation with the usual boundary values and initial value $u(x, 0) = f(x)$ is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

  where the c_n are the coefficients of the Fourier sine series of f—namely

$$c_n = \frac{\langle f(x), \sin\left(\frac{n\pi}{L}x\right)\rangle}{\langle \sin\left(\frac{n\pi}{L}x\right), \sin\left(\frac{n\pi}{L}x\right)\rangle} = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx.$$

  (The $\frac{2}{L}$ just comes from $\langle \sin\left(\frac{n\pi}{L}x\right), \sin\left(\frac{n\pi}{L}x\right)\rangle$.)

Most important idea today: How to solve the heat equation with the usual boundary values and any "nice enough" initial value.