Review

1. Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is an orthogonal basis for an inner product space V.
   (a) If $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$, what is $\langle \mathbf{u}, \mathbf{v}_i \rangle$?
   (b) Let $\mathbf{w} \in V$. Express $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$. [Hint: if you don't remember how to do this, try looking at part (a).]

The Heat Equation

1. Find a solution to the following differential equation.

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}; \quad u(0,t) = u(L,t) = 0; \quad u(x,0) = \sin(\pi x/L) + 13\sin(5\pi x/L)$$

Fourier Series

1. Find the Fourier series of the function $f\colon [-\pi, \pi] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{\pi}{2} + x \text{ when } x \leq 0 \\ \frac{\pi}{2} - x \text{ when } x > 0 \end{cases}$$

2. Find the Fourier series of the function $f\colon [-\pi, \pi] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} -1 \text{ when } x \leq 0 \\ 1 \text{ when } x > 0 \end{cases}$$

3. Suppose the Fourier series of an infinitely differentiable function f is

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \cos(nx)$$

   (so the coefficients of the sine terms are zero).
   (a) What is $f(0)$?
   (b) What is $f(\pi)$?
   (c) What are the fourier coefficients of $f(x)\cos(x)$?

Definitions and Theorems

Definitions:

- Inner product space, orthogonal basis, orthogonal projection (these are not new, but they are important for today)

- Fourier Series, fourier coefficients
- Periodic function

Theorems:

- The functions $\cos(0x) = 1, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots$ are orthogonal on the inner product space $C^\infty([-\pi, \pi])$ with inner product given by $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$.

  In fact, it is often more useful to instead say that the set $\{e^{inx} \mid n \in \mathbb{Z}\}$ is an orthogonal set in this inner product space, but we will not talk about that in this class since we never said how inner products should work when complex numbers are involved.

- If f is a continuous function on $[-\pi, \pi]$ whose derivative is piecewise continuous and $f(-\pi) = f(\pi)$, then the fourier series for f converges to f at every point.

Most important idea today: Thanks to orthogonality, it is easy to write any "nice-enough" function using sines and cosines.