Review

For each item below, explain why it is true or provide a counterexample to show it is false.

1. It is not possible to find five vectors in $\mathbb{R}^3$ that do not span $\mathbb{R}^3$.
2. If $m < n$ then it is not possible for m vectors to span all of $\mathbb{R}^n$.
3. If $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are vectors in $\mathbb{R}^n$ then $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ contains either one vector or infinitely many vectors.

Challenge problem: What is $\text{span}\{\mathbf{u}_1, \mathbf{u}_2\} \cap \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$?

$$\mathbf{w} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

Linear Independence

1. Prove that each of the following lists of vectors is linearly dependent.

   (a) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 17 \\ -3 \end{bmatrix}$

   (c) $\mathbf{u}, \mathbf{v}, 3\mathbf{u} - 4\mathbf{v}$ where $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^4$.

2. Which of the following lists of vectors are linearly dependent?

   (a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

   (b) $\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

   (c) $\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

3. Can you think of a general method to check if a list of vectors is linearly dependent?
4. Is it possible to find four vectors in $\mathbb{R}^3$ that are not linearly dependent?

Matrices

1. For each of the following, either calculate the product of the matrix and the vector or state that the product is not defined.

(a) $\begin{bmatrix} -5 & 4 & 7 \\ 8 & 9 & 1 \\ 0 & -1/2 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 7 & -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$

(g) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(h) $\begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

2. Show that if A is an $n \times m$ matrix, $\mathbf{v}$ is a vector in $\mathbb{R}^m$ and c is a real number then $A(c\mathbf{v}) = c(A\mathbf{v})$.