Review

1. Find a solution to the following initial value problem.

$$y'' + 2y' + 5y = 26e^{2t}; \quad y(0) = 3; \quad y'(0) = 9$$

Wronskian

1. (a) Find the Wronskian of 1 and e^{t^2}.
   (b) Are 1 and e^{t^2} linearly independent? (Hint: Use part (a))
   (c) Is there any linear ODE for which both 1 and e^{t^2} are solutions?
2. (a) Find the Wronskian of $t|t|$ and t^2.
   (b) Are $t|t|$ and t^2 linearly independent?

Systems of ODEs

1. Reduce the following higher order ODE to a system of first order ODEs and then put that system in normal form.

$$y''' + e^t y'' - \cos(t)y = 17$$

2. Reduce the following system of higher order ODEs to a system of first order ODEs and then put that system in normal form.

$$\begin{aligned} y'' &= 5y' - 6z' + z + \sin(t) \\ z'' &= y' + z + 2 \end{aligned}$$

3. Find the derivative of the following vector-valued functions.

   (a) $$\mathbf{f}(t) = \begin{bmatrix} \sin(t) \\ t^2 + te^{5t} \end{bmatrix}$$

   (b) $$\mathbf{y}(t) = e^{5t}\mathbf{v} \text{ where } \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

4. Check if each function given below is a solution to $\mathbf{y}' = A\mathbf{y}$.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$$

   (a) $$\mathbf{f}(t) = e^{3t} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$$

   (b) $$\mathbf{g}(t) = \begin{bmatrix} \sin(t) \\ 2 \\ 3e^{5t} \end{bmatrix}$$

5. Suppose $X(t)$ is a fundamental matrix for the system $\mathbf{y}' = A\mathbf{y}$. Solve the initial value problem $\mathbf{y}' = A\mathbf{y}; \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$X(t) = \begin{bmatrix} e^{3t} & 2e^{7t} \\ 5e^{3t} & 7e^{7t} \end{bmatrix}$$

Definitions and Theorems

Definitions:

- Pre-Wronskian
- Wronskian
- System of ODEs
- Vector of functions (equivalently a function from $\mathbb{R}$ to $\mathbb{R}^n$)
- Normal Form
- Fundamental matrix

Theorems:

- The Wronskian Lemma: Suppose $y_1, \ldots, y_n$ are solutions to a linear ODE. Then the Wronskian $W[y_1, \ldots, y_n]$ is nonzero everywhere if $y_1, \ldots, y_n$ are linearly independent and otherwise it is zero everywhere.

  Caution: This is not true if $y_1, \ldots, y_n$ are not all solutions to the same linear ODE. For arbitrary functions, if the Wronskian is nonzero at any point then they are linearly indepdent, but there *are* linearly indepdent functions whose Wronskian is zero everywhere.
- The initial value problem for a system of first order linear ODEs always has a solution and that solution is always unique.
- (If we have time) If $\mathbf{u}$ is an eigenvector of A with eigenvalue r then $\mathbf{y}(t) = e^{rt}\mathbf{u}$ is a solution to $\mathbf{y}'(t) = A\mathbf{y}(t)$.

Most important idea today: Every higher order linear ODE can be reduced to a system of first order ODEs.

Most important idea today if we have time to get to it: To find solutions to $\mathbf{y}'(t) = A\mathbf{y}(t)$, find eigenvectors of A. Moral: any time you see a linear transformation, its eigenvectors are probably important!!