Review

1. Write a homogeneous differential equation such that $t^2e^{3t} + 5$ is a solution.
2. Write a homogeneous differential equation such that $e^{-2t}\cos(4t) + e^t$ is a solution.

Nonhomogeneous ODEs

1. Find a solution to the following ODEs.
   (a) $y'' - y' + y = 4t^2 + 8$
   (b) $2y'' - y' = 5\cos(t)$
   (c) $y'' - 6y' + 9y = e^{3t}$
2. You may assume without checking that t^3e^{-t} is a solution to $y''' + 3y'' + 3y' + y = 6e^{-t}$, that $\sin(t)$ is a solution to $y''' + 3y'' + 3y' + y = -2\sin(t) + 2\cos(t)$. Find a solution to the following ODEs.
   (a) $y''' + 3y'' + 3y' + y = e^{-t}$
   (b) $y''' + 3y'' + 3y' + y = e^{-t} + \sin(t) - \cos(t)$
3. Find the general solution to the following ODEs.
   (a) $2y'' - y' = 5\cos(t)$.
   (b) $y'' + 2y' + 5y = 26e^{2t}$.
4. Find a solution to the following initial value problems.
   (a) $2y'' - y' = 5\cos(t); \quad y(0) = 3; \quad y'(0) = 0.$
   (b) $y'' + 2y' + 5y = 26e^{2t}; \quad y(0) = 3; \quad y'(0) = 9.$

Wronskian

1. (a) Find the Wronskian of 1 and e^{t^2}.
   (b) Are 1 and e^{t^2} linearly independent? (Hint: Use part (a))
   (c) Is there any linear ODE for which both 1 and e^{t^2} are solutions?
2. (a) Find the Wronskian of $t|t|$ and t^2.
   (b) Are $t|t|$ and t^2 linearly independent?

Definitions and Theorems

Definitions:

- Pre-Wronskian
- Wronskian

Theorems:

- If y_1 is a solution to the ODE $a_n y^{(n)} + \ldots + a_1 y' + a_0 y = f_1$ and y_2 is a solution to the ODE $a_n y^{(n)} + \ldots + a_1 y' + a_0 y = f_2$ then $c_1 y_1 + c_2 y_2$ is a solution to $a_n y^{(n)} + \ldots + a_1 y' + a_0 y = c_1 f_1 + c_2 f_2$. The textbook calls this the "superposition principle" but it is really just part of the definition of 'linear transformation.'

- If y_p is a solution to the ODE $a_n y^{(n)} + \ldots + a_1 y' + a_0 y = f$ and the general solution of the homogeneous ODE $a_n y^{(n)} + \ldots + a_1 y' + a_0 y = 0$ is $c_1 y_1 + \ldots + c_n y_n$ then $y_p + c_1 y_1 + \ldots + c_n y_n$ is the general solution to $a_n y^{(n)} + \ldots + a_1 y' + a_0 y = f_1$. This is really just a statement about linear transformations that we first saw in chapter 1, section 5 of the linear algebra textbook.

- (If we have time) The Wronskian Lemma: Suppose $y_1, \ldots, y_n$ are solutions to a linear ODE. Then the Wronskian $W[y_1, \ldots, y_n]$ is nonzero everywhere if $y_1, \ldots, y_n$ are linearly independent and otherwise it is zero everywhere.

  Caution: This is not true if $y_1, \ldots, y_n$ are not all solutions to the same linear ODE. For arbitrary functions, if the Wronskian is nonzero at any point then they are linearly indepdent, but there *are* linearly indepdent functions whose Wronskian is zero everywhere.

Most important idea today: If T is a linear transformation then the set of solutions to $T(\mathbf{x}) = \mathbf{b}$ is just the kernel of T translated by some vector and therefore to find all solutions to a nonhomogeneous linear ODE it is enough to find one solution to the nonhomogeneous ODE and all solutions to the corresponding homogeneous ODE.