Spectral Theorem

1. Is the matrix A orthogonally diagonalizable? If so, find a diagonal matrix D and an orthogonal matrix P such that $A = PDP^T$.

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

2. Suppose A is a symmetric matrix. Show that A is positive semi-definite if and only if all eigenvalues of A are nonnegative.

3. Show that if A is any matrix, $A^T A$ is symmetric and positive semi-definite.

4. (This one's kind of tricky.) Show that if A is a symmetric matrix then the largest eigenvalue of A is the maximum possible value of

$$\frac{\mathbf{x} \cdot (A\mathbf{x})}{\mathbf{x} \cdot \mathbf{x}}$$

(Hint: what happens to the expression above if you pick an orthonormal basis of eigenvectors for A and try writing some $\mathbf{x}$ in terms of that basis?)

Definitions and Theorems

Definitions and examples:

- Symmetric matrix. Also known as self-adjoint (especially in math) or Hermitian (especially in physics) although these mean something slightly different than symmetric when there are complex numbers involved.
- Quadratic Form.
- Positive definite matrix, positive semi-definite matrix (PSD).

Theorems:

- The Spectral Theorem: If A is an $n \times n$ symmetric matrix then A has only real eigenvalues and there is an orthogonal basis for $\mathbb{R}^n$ consisting of eigenvectors of A (i.e. A is orthogonally diagonalizable).

Most important idea today: THE SPECTRAL THEOREM! (Having a basis of orthogonal eigenvectors is super useful.)