Review

1. Is the following matrix invertible? Try to answer without doing any calculations.

$$\begin{bmatrix} 1 & 5 & 1 & 2 \\ 2 & 16 & 2 & 54 \\ 3 & -7 & 3 & 13 \\ 4 & 0 & 4 & -30 \end{bmatrix}$$

2. Suppose A is an $n \times m$ matrix and B is an $m \times p$ matrix.
   (a) Show that $\dim(\operatorname{Null} B) \leq \dim(\operatorname{Null} AB)$.
   (b) Show that $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$. (Hint: how does the rank of a matrix relate to the dimension of its null space?)
   (c) Show that $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$.
   (d) Is it always true that $\dim(\operatorname{Null} B) = \dim(\operatorname{Null} AB)$?

Change of Basis

1. Let $\mathcal{B} = \{x+1, x^2+x, x^2+1\}$ and $\mathcal{C} = \{x^2+x+1, x^2, x\}$. Both are bases for $\mathbb{P}_2$ (you do not have to check this). Suppose that

$$[p]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

   What is $[p]_{\mathcal{C}}$?

2. With $\mathcal{B}$ and $\mathcal{C}$ as in the previous problem, find the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$.

3. Suppose V is a vector space and $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}, \mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are two different bases for V. If $\mathbf{w} = 3\mathbf{u}_1 - \mathbf{u}_2 + \mathbf{u}_3$ and the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$, $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$, is as given below, what is $[\mathbf{w}]_{\mathcal{C}}$?

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

4. If $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are bases for a vector space, and $\mathbf{v}$ is a vector in that vector space, what is $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1}[\mathbf{v}]_{\mathcal{C}}$? What is $\underset{\mathcal{D}\leftarrow\mathcal{C}}{P}\,\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{v}]_{\mathcal{B}}$?

5. Suppose $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

   Find the matrix of T relative to the basis $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$.

6. With T as in the previous question, find the standard matrix of T.

7. If A and B are similar matrices and C and D are similar matrices then are AC and BD similar matrices?