Review

1. Find four different subspaces of $\mathbb{R}^3$.

2. What is $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}^2$? What about $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}^4$?

3. **Challenge Problem:** Find a formula for $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}^n$.

Vector Spaces

1. Which of the following are vector spaces?

   (a) The set of polynomials with real coefficients of degree exactly 3

   (b) The set of 2×3 matrices in RREF

   (c) The set of 5×5 matrices X such that $AX = 0$, where A is a 5×5 matrix.

   (d) The set of differentiable functions $f \colon \mathbb{R} \to \mathbb{R}$ such that $f' = f$ (where f' means the derivative of f)

   (e) The set of even functions from $\mathbb{R}$ to $\mathbb{R}$ (i.e. the set $\{f \colon \mathbb{R} \to \mathbb{R} \mid f(x) = f(-x) \text{ for all } x \in \mathbb{R}\}$)

   (f) The set of convergent sequences of real numbers whose limit is 0.

   (g) The set of convergent sequences of real numbers whose limit is 1.

   (h) The set of English words.

2. Of the items in the previous question that are vector spaces, are any of them subspaces of some other vector spaces? If so, which ones?

3. Find 3 subspaces of $\mathbb{P}_3$.

4. Answer the following questions.

   (a) Are the polynomials $1, x^2, 3x^2 - 2$ linearly independent?

   (b) Are the functions $\sin^2(x), \cos^2(x)/2, 1$ linearly independent?

   (c) Do the functions $f \colon [0,1] \to \mathbb{R}$ and $g \colon [0,1] \to \mathbb{R}$ defined by $f(x) = x$ and $g(x) = \sin(x)$ span all of $C([0,1])$? (Hint: think about $f(0)$ and $g(0)$.)

   (d) Is the sequence $(1, 0, 1, 0, \ldots)$ in the span of $(1, 1, 1, 1, \ldots)$ and $(1, -1, 1, -1, \ldots)$?

   (e) Are the following matrices linearly independent?

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

5. Answer the following questions.

   (a) Is $\{1, x, x^2\}$ a basis for $\mathbb{P}_2$?

   (b) Is $\{1, x, x^2\}$ a basis for $\mathbb{P}_3$?

(c) Is $\{1, x^2, x, 3x^2 - 2\}$ a basis for $\mathbb{P}_2$?

(d) With f and g as in part (c), is $\{f, g\}$ a basis for $C([0,1])$?

(e) What is the dimension of $\mathbb{P}_2$?

6. Which of the following are linear transformations?

(a) $T\colon \mathbb{P}_3 \to \mathbb{P}_3$ defined by $T(p) = \frac{dp}{dx}$.

(b) $T\colon M_{2\times 3} \to M_{4\times 3}$ defined by $T(B) = AB$ where

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$$

(c) $T\colon M_{2\times 2} \to M_{2\times 2}$ defined by $T(A) = A + I$.

(d) $T\colon C([0,1]) \to C([0,1])$ defined by $T(f) = f^2$.

(e) $T\colon C(\mathbb{R}) \to \mathbb{R}^2$ defined by $T(f) = \begin{bmatrix} f(0) \\ f(\pi) \end{bmatrix}$.

(f) $T\colon C([0,1]) \to \mathbb{R}$ defined by $T(f) = \int_0^1 f(x)\,dx$.