Basic Mathematical Concepts and Terminology

Mathematicians have their own way of talking and their own way of writing things. This is not because mathematicians are crazy, but because they have found this language helpful when learning and doing math. In this document we will review some of the language and notation that mathematicians use.

It may not be necessary for you to read this entire document. Use it as a reference while doing the first homework assignment.

0.1 Some Common Sets

First we review some notation that may already be familiar to you:

- $\mathbb{N}$ is the set of natural numbers: $\{0, 1, 2, \ldots\}$.
- $\mathbb{Z}$ is the set of integers: $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
- $\mathbb{R}$ is the set of real numbers. It includes all the integers, as well as numbers like 5.4, -123.09445, and π.
- $\mathbb{R}^n$ is the set of lists of n real numbers. For instance, $\mathbb{R}^3$ contains elements like

$$\begin{bmatrix} 1 \\ -3 \\ 2.5 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

  which are sometimes also written as $(1, -3, 2.5)$ and $(1, 1, 1)$. It is common to picture $\mathbb{R}^2$ as the plane (and probably the way in which you are probably most used to picturing it) but it is always possible to just think of it as all lists with two numbers.

1 Sets

It is common in mathematics to talk about **sets**. A set is just a collection of objects. If A is a set then we write $x \in A$ to indicate that x is one of the objects in A and we read this as "x is an **element** of A." Similarly, we write $x \notin A$ to indicate that x is not one of the objects in A.

Example 1:

- $1 \in \mathbb{N}$
- $-1 \notin \mathbb{N}$

1.1 Ways to Describe Sets

There are several ways to describe a set:

(1) One is to just explicitly list out the elements (enclosed by curly brackets).

Example 2: $\{1, 4, -2, 0\}$ refers to the set that contains 1, 4, -2, and 0. The order the elements are written in does not matter—e.g. $\{1, 4, -2, 0\} = \{-2, 0, 1, 4\}$. Also, sets do not have to contain numbers. For instance, $\{\text{Obama}, \text{Reagan}\}$ is also a set (which contains former US presidents).

(2) For some sets it is unreasonable to write out every single element in the set. So it is usually acceptable to list enough elements to make it clear what the set contains and then use dots to indicate the rest of the elements.

Example 3: For instance $\{0, 2, 4, 6, \ldots\}$ is an acceptable way to indicate the set of all even natural numbers. Similarly $\{1, 2, 3, \ldots, 10\}$ is an acceptable way to indicate the set of all integers between 1 and 10.

Sometimes this method of describing a set may be amibiguous and it is better to describe it some other way. For instance $\{1, 3, \ldots\}$ could either mean all odd natural numbers or all powers of three. **Try to avoid this kind of ambiguity.**

(3) It is also okay to just give an unambiguous English description of what a set contains.

Example 4: The description "the set of all numbers greater than 5" is an acceptable way to define a set.

(4) If you want to define the set of set of objects with some property P, it is common to write $\{x \mid x \text{ has property } P\}$ (everything after the vertical line is the description of things in the set).

Example 5: $\{n \mid n \in \mathbb{N} \text{ and } n > 5\}$ defines the same set as in the previous example. It is also common to write this as $\{n \in \mathbb{N} \mid n > 5\}$.

(5) There are a few special ways to describe certain types of sets.

Example 6:

- $[0, 1]$ refers to the set of real numbers between 0 and 1 including both 0 and 1.
- $[2, 4)$ refers to the set of real numbers between 2 and 4 including 2 but not including 4.
- $\varnothing$ refers to the set with nothing at all in it, often called the "empty set."

1.2 Things You Can Do with Sets

We have already seen that $x \in A$ means "the set A contains x." But this is not the only thing we might want to say about sets. For instance, $\mathbb{N} \notin \mathbb{Z}$ since $\mathbb{Z}$ only contains numbers and $\mathbb{N}$ is not a number. But there is still an important relationship between $\mathbb{N}$ and $\mathbb{Z}$. In particular, every element of $\mathbb{N}$ is also an element of $\mathbb{Z}$.

This relationship is denoted by $\mathbb{N} \subseteq \mathbb{Z}$, read as "$\mathbb{N}$ is a **subset** of $\mathbb{Z}$." In general, $A \subseteq B$ if every element of A is an element of B and $A \nsubseteq B$ otherwise.

Example 7:

- $\mathbb{N} \subseteq \mathbb{R}$, $\mathbb{R} \nsubseteq \mathbb{Z}$.
- $\{0, 1, 5\} \subseteq \mathbb{N}$
- Every set is a subset of itself: $\mathbb{N} \subseteq \mathbb{N}$, $\{\text{Obama}, \text{Regan}\} \subseteq \{\text{Obama}, \text{Reagan}\}$ and so on.
- For any set A, $\varnothing \subseteq A$ (why is this?).

If A and B are sets with no elements in common then we say A and B are **disjoint**. For instance $\{-1, -2\}$ and $\mathbb{N}$ are disjoint, but $\mathbb{N}$ and $\mathbb{Z}$ are not.

Here are three useful things you can do with sets, which you may already be familiar with. If A and B are sets then:

- The **intersection** of A and B, written $A \cap B$ is the set containing everything that is an element of both A and B.
- The **union** of A and B, written $A \cup B$ is the set containing everything that is either in A or B or both.
- The **set difference** of A and B, written $A \setminus B$ is the set containing everything that is in A but not B.

Example 8: If $A = \{0, 1, 2, 3\}$ and $B = \{1, 3, 5, 7\}$ then

- $A \cap B = \{1, 3\}$.
- $A \cup B = \{0, 1, 2, 3, 5, 7\}$
- $A \setminus B = \{0, 2\}$.

2 Functions

You are probably already familiar with functions on the real numbers. For instance you might have seen phrases like "the function $f(x) = x^2 + 2$ is continuous." But it is very common in mathematics—and it will sometimes be useful in this class—to consider functions defined on sets that are not the real numbers.

In general, a function is just something that takes inputs in one set and for each input, gives an output in some other set (which might be the same as the first set). The set of inputs is called the **domain** and the set of possible outputs is called the **codomain**. The domain of a function f is denoted by $\text{dom}(f)$. I don't know of any common notation to denote the codmain of a function. If we want to express that f is a function with domain A and codomain B we usually write

$$f : A \to B.$$

2.1 Ways to Describe Functions

Just as with sets, there are several ways to describe a function. Examples of each of these are given below.

(1) Explicitly list out what output each input gets sent to.

(2) Give some formula that can be used to find what output each input gets sent to.

(3) Describe using an unambiguous English sentence what output each input gets sent to.

(4) Give several different ways to find what output an input gets sent to, each of which is valid for only part of the domain. This is called **definition by cases**.

Here are some examples of the ways of describing functions listed above.

Example 9:

(1) $f\colon \{0,1,2\} \to \{\text{Obama}, \text{Reagan}\}$ defined by $f(0) = \text{Obama}$, $f(1) = \text{Reagan}$, and $f(2) = \text{Obama}$. Here we have just listed out every value for the function f.

(2) $g\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 3$. This means that for each $x \in \mathbb{R}$, the function f on input x gives output $x^2 + 3$.

(3) $h\colon \{0, 1, \ldots, 10\} \to \mathbb{N}$ defined by $g(n) =$ "the smallest natural number with at least n letters when written in English." Here, for instance, $g(5) = 3$ because 3 has five letters, but 0, 1, and 2 have less than five letters (in English).

(4) $k\colon \mathbb{Z} \to \mathbb{N}$ defined by

$$h(n) = \begin{cases} 2^n & \text{if } n \geq 0 \\ 3^{-n} & \text{if } n < 0. \end{cases}$$

In this case, $h(3) = 8$ and $h(-2) = 9$.

When defining a function, **it is important that your description says unambiguously what output every input gets sent to and that each output is actually in the codomain.**

Below are a few examples of definitions of functions that do not make sense.

Example 10: The sentence: Let $f\colon \mathbb{Z} \to \mathbb{N}$ be the function defined by

$$f(n) = \begin{cases} 1 & \text{if } n > 0 \\ -1 & \text{if } n < 0 \end{cases}.$$

does not make sense because it does not say what output the input 0 gets sent to (i.e. it does not say what $f(0)$ is).

Example 11: The sentence: "Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1/x$" does not make sense because $1/0$ is undefined. So it is not clear what output the input 0 gets sent to. Saying "let $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be defined by $f(x) = 1/x$" does make sense.

Example 12: The sentence "Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$" does not make sense because $\sqrt{-1}$ is not an element of $\mathbb{R}$. Saying "let $f\colon \mathbb{R} \to \mathbb{C}$ be the function defined by $f(x) = \sqrt{x}$" does make sense.

Also note that the domain and codomain of a function are part of its definition. So for instance the function $f\colon \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x^2$ is not equal to the function $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$.

In this class, we will occasionally see functions that are a little stranger than the ones you are used to seeing.

Example 13: Let $f\colon \{g \mid g \text{ is a function from } \mathbb{R} \text{ to } \mathbb{R}\} \to \mathbb{R}$ be defined by $f(g) = g(5)$.

2.2 Things You Can Say About Functions

The following are two important properties that a function can have. A function f is

- **One-to-one** if for every element y of the codmain there is *at most* one element of the domain that gets sent to y.
- **Onto** if for every element y of the codomain there is *at least* one element of the domain that gets sent to y.

The **range** of a function $f\colon A \to B$, written range(f), is the set of outputs that some input *actually* gets sent to. More formally,

$$\text{range}(f) = \{y \in B \mid \text{there is some } x \in A \text{ such that } f(x) = y\}.$$

Note that range$(f) \subseteq B$ but range(f) and B are not always equal (they are equal exactly when f is onto—can you see why?)

Example 14: For the functions in example 9:

(1) f is not one-to-one, but it is onto. It is not one-to-one because two elements of the domain—namely 0 and 2—get sent to Obama. In other words, it is not one-to-one because $f(0) = f(2)$.

(2) g is neither one-to-one nor onto. It is not one-to-one because $g(2) = g(-2) = 7$. It is not onto because no input gets sent to -1. In other words, -1 is not in the range.

(3) h is neither one-to-one nor onto. It is not one-to-one because $g(0) = g(1) = 0$ (because 0 has four letters in English). It is not onto because 300 is not in the range (since 300 has twelve letters in English).

(4) k is one-to-one, but it is not onto. It is not onto because 5 is not in the range (since it is neither a power of 2 nor a power of 3).

(5) The function from example 13 is not one-to-one, but it is onto (why?)

Suppose a function is both one-to-one and onto. Then given any element of the codomain there is exactly one element of the domain that gets sent to it. In other words, we know how to "undo" the function. More precisely, say that $f\colon A \to B$ is one-to-one and onto. Define $f^{-1}\colon B \to A$ by

$$f^{-1}(x) = \text{ the unique element of } A \text{ that gets sent to } x \text{ by } f.$$

This function is called the **inverse** of f and functions that are both one-to-one and onto are often called **invertible**. Notice that the inverse of the inverse is just the original function.

Example 15 (Functions that are both one-to-one and onto):

- The function $T\colon \mathbb{R} \to \mathbb{R}$ defined by $T(x) = 2x+3$ is both one-to-one and onto. Its inverse is the function $T^{-1}\colon \mathbb{R} \to \mathbb{R}$ defined by $T^{-1}(x) = (x-3)/2$ (why?)
- The function $f\colon \mathbb{N} \to \{\text{powers of two}\}$ defined by $f(n) = 2^n$ is one-to-one and onto. Its inverse is the function $f^{-1}\colon \{\text{powers of two}\} \to \mathbb{N}$ defined by $f^{-1}(n) = \log_2 n$.
- For every set A there is a function $\mathrm{id}_A\colon A \to A$ defined by $\mathrm{id}_A(x) = x$, called the **identity function on A** (think of like the "do-nothing function"). It is always one-to-one and onto and it is always its own inverse.

2.3 Useful Things You Can Do with Functions

If $f\colon A \to B$ and $g\colon B \to C$ are functions then there is a way we can combine f and g to get a new function with domain A and codomain C. This function, called the **composition** of g and f and written $g \circ f\colon A \to C$, works like this: starting with an input from the set A, the function f gives an output in the set B. We then treat this output as an input to g, which gives us an output in the set C. More formally, $g \circ f$ is defined by

$$g \circ f(x) = g(f(x)).$$

In slogan form "first do f then do g."

Example 16: Let $f\colon \mathbb{N} \to \mathbb{N}$ be the function defined by $f(n) = n+1$ and let $g\colon \mathbb{N} \to \mathbb{N}$ be the function defined by $g(n) = 3n$. Then $g \circ f\colon \mathbb{N} \to \mathbb{N}$ is the function defined by

$$g \circ f(n) = g(f(n)) = g(n+1) = 3(n+1) = 3n+3.$$

Notice that this is *not* equal to $f \circ g$, which is the function from $\mathbb{N}$ to $\mathbb{N}$ defined by $f \circ g(n) = 3n+1$.

It is important that the codomain of f is equal to the domain of g.

Example 17: Let $f\colon \mathbb{N} \to \mathbb{R}$ be the function defined by $f(n) = \sqrt{n}$ and let $g\colon \mathbb{R} \to \mathbb{R}^2$ be the function defined by

$$g(x) = \begin{bmatrix} x \\ -x \end{bmatrix}.$$

Then $g \circ f\colon \mathbb{N} \to \mathbb{R}^2$ is the function defined by

$$g \circ f(n) = g(f(n)) = g\left(\sqrt{n}\right) = \begin{bmatrix} \sqrt{n} \\ -\sqrt{n} \end{bmatrix}.$$

On the other hand, $f \circ g$ does not make any sense because if $x \in \mathbb{R}$ then $g(x)$ is in $\mathbb{R}^2$ and it doesn't make sense to apply f to an element of $\mathbb{R}^2$.

Here is another useful thing you can do with functions. Say that $f\colon A \to B$ is a function and x is an element of B. There are many situations (and we will see some later in this class) in which it is useful to know what elements of A—if any—get sent to x by f. More generally, if C is a subset of B, it is sometimes useful to know what elements of A get sent by f to *some* element of C. The notation $f^{-1}(C)$, called the **inverse image** of C, is used to denote the set of such elements. More formally,

$$f^{-1}(C) = \{x \in A \mid f(x) \in C\}.$$

This is not the same as the inverse of a one-to-one and onto function. For one thing, we can talk about the inverse image even if the function is not one-to-one and onto. For antoher thing, the inverse image takes a *subset* of the codomain and gives you a *subset* of the domain while the inverse of a function takes an *element* of the codomain and gives you an *element* of the domain. However, they are related: if $f\colon A \to B$ is one-to-one and onto then $f^{-1}(\{x\})$ always has exactly one element, which is always $f^{-1}(x)$.

Example 18: Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Then $f^{-1}([25, 49]) = [-7, -5] \cup [5, 7]$ (recall that $[a, b]$ refers to all real numbers between a and b, including a and b).