Midterm 2 Review Questions

Disclaimer: This worksheet is intended to remind you of the major topics covered after the first midterm. It does not cover topics from before the first midterm even though you may need to understand them to solve problems on the second midterm. It also does not cover each topic thoroughly or include every concept that we've learned in class since that would require a much longer worksheet.

One of the biggest ideas in linear algebra: You can completely determine a linear transformation by saying what it does to each basis vector in some basis.

Computations that you should be able to do:

- Find the coordinate vector of a vector relative to a basis
- Find the matrix of a linear transformation relative to bases for its domain and codomain
- Find the change of basis matrix for two bases for $\mathbb{R}^n$
- Find the eigenvalues and eigenvectors of a matrix (including complex eigenvalues)
- Compute large powers of a diaognalizable matrix
- Take dot products and find distances between vectors in $\mathbb{R}^n$
- Find the orthogonal projection of a vector onto a subspace of $\mathbb{R}^n$
- Use Gram-Schmidt to turn a basis for a subspace into an orthogonal basis
- Find the least squares solution to a system of linear equations

Other things you should know how to do:

- Check if a subset of a vector space is a subspace
- Check if a function between vector spaces is a linear transformation
- Check if vectors in a vector space are linearly independent, span the whole space, etc by picking a basis for the vector space, finding coordinate vectors relative to that basis and doing row reduction
- Check if two matrices are similar in some cases (you do not know a general algorithm to tell if two matrices are similar because it requires knowledge not covered in this class)
- Write 2×2 real matrices with complex eigenvalues in terms of a scaling multiplied by a rotation matrix
- Check if two vectors are orthogonal

Fundamental definitions you should know by heart:

- Linear transformation between vector spaces
- Basis of a vector space
- Eigenvector of a linear transformation
- Similar matrices
- Orthogonal vectors
- Projection of a vector onto a subspace

1. Which of the following are vector spaces?
   (a) The set of differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f' = f$ (where f' means the derivative of f).
   (b) The set of 3×3 matrices whose determinant is 1.
   (c) The set of polynomials whose degree is an even number.
   (d) The set of vectors in $\mathbb{R}^3$ whose norm is at most 1.

2. What is the dimension of the subspace $\operatorname{span}\{1 + x, x + x^2, x^2 - 1\}$ of $\mathbb{P}_2$?

3. For this problem, you may assume without checking that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ shown below are linearly independent.

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 3 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

   (a) Suppose that $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ is a linear transformation such that

$$\begin{aligned} T(\mathbf{v}_1) &= 2\mathbf{v}_1 \\ T(\mathbf{v}_2) &= \mathbf{v}_1 + 2\mathbf{v}_2 \\ T(\mathbf{v}_3) &= 3\mathbf{v}_3 \\ T(\mathbf{v}_4) &= \mathbf{v}_1 \end{aligned}$$

   Find a basis $\mathcal{B}$ and a matrix A such that A is the matrix of T in the basis $\mathcal{B}$—i.e. such that ${}_{\mathcal{B}}[T]_{\mathcal{B}} = A$.
   (b) Find an invertible matrix P such that PAP^{-1} is the standard matrix of T. You do not need to show that the matrix you find is invertible and you do not need to find its inverse.

4. Give an example of a square matrix A and a scalar λ such that λ^2 is an eigenvalue of A^2 but λ is not an eigenvalue of A, or explain why no such example exists.

5. Let $V = \operatorname{span}\{\sin(x), \cos(x)\}$ and let $T\colon V \to V$ be the function defined by $T(f) = f'$ (i.e. $T(f)$ is the derivative of f). Find the eigenvalues of T.

6. Ted is a math 54 student who is given the following problem on an exam:

> Find $\operatorname{proj}_W(\mathbf{x})$ where $W = \operatorname{span}\{\mathbf{u}, \mathbf{v}\}$ and
>
> $$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

   Ted attempts to solve this problem by calculating

$$\operatorname{proj}_W(\mathbf{x}) = \frac{\mathbf{u} \cdot \mathbf{x}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} + \frac{\mathbf{v} \cdot \mathbf{x}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} = \frac{7}{14}\mathbf{u} + \frac{3}{3}\mathbf{v} = \begin{bmatrix} 3/2 \\ 0 \\ 5/2 \end{bmatrix}.$$

   What is wrong with Ted's calculation (hint: it's not an arithmetic mistake)? What should he do to correctly solve the problem?