

Midterm 1 Review Questions

1. **Conceptual Question.** Suppose that A is an $m \times n$ matrix. Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ denote the columns of A and let a_{ij} denote entry (i, j) of A (so the i^{th} entry of \mathbf{v}_j is a_{ij}). Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ denote the linear transformation defined by $T(\mathbf{v}) = A\mathbf{v}$. Also suppose that \mathbf{b} is a vector in \mathbb{R}^m with entries b_1, b_2, \dots, b_m .

For each part below, express each statement in the languages of pivots, systems of linear equations, matrix equations, span and linear independence of vectors, and linear transformations. The first part has been done for you.

- (a) $[A \mid \mathbf{b}]$ has no pivot in the rightmost column when put into REF. This is equivalent to each of the following.

$$\bullet \begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{cases} \text{ is consistent}$$

- $A\mathbf{x} = \mathbf{b}$ has a solution
- $\mathbf{b} \in \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$
- \mathbf{b} is in the range of T

- (b) $A\mathbf{x} = \mathbf{b}$ has a unique solution.
 (c) The vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent.
 (d) T is onto.

2. **Computational Question.**

- (a) For what values of c is the linear transformation given below one-to-one? For what values of c is it onto?

$$T \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 + x_2 + 5x_3 \\ 2x_1 + 4x_3 \\ 3x_1 + 6x_3 \\ x_1 + x_2 + cx_3 \end{bmatrix}$$

- (b) For what values of c is the solution are the following vectors linearly independent?

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 5 \\ 4 \\ 6 \\ c \end{bmatrix}$$

- (c) For what values of c does the following matrix equation have a unique solution?

$$\begin{bmatrix} 1 & 1 & 5 \\ 2 & 0 & 4 \\ 3 & 0 & 6 \\ 1 & 1 & c \end{bmatrix} \mathbf{x} = \mathbf{0}$$

3. **Conceptual Question.** Check whether each function below is a linear transformation. If so, find its standard matrix.

- (a) The function $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}$.
- (b) The function $S: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+3 \\ y \end{bmatrix}$.

4. **Computational Question.**

- (a) Find all solutions to the following system of linear equations.

$$\begin{aligned} x_1 + 2x_2 + 4x_4 &= 0 \\ 2x_1 + 4x_2 + 5x_3 - 3x_4 &= 0 \\ 5x_1 + 10x_2 + 20x_4 &= 0 \end{aligned}$$

- (b) Write the set of solutions to $A\mathbf{x} = \mathbf{b}$ in parametric form, where

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & 5 & -3 \\ 5 & 10 & 0 & 20 \end{bmatrix}$$

- (c) Find a basis for $\text{Null } A$ (with A as above)
- (d) Find a basis for $\text{Col } A$ (with A as above)

5. **Computational Question.** Find the determinant of the following matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 5 \\ 0 & -1 & -6 \end{bmatrix}$$

6. **Conceptual Question.** True or false: If A is an $n \times m$ matrix, then the set of solutions to $A\mathbf{x} = \mathbf{0}$ is a subspace of \mathbb{R}^m .
7. **Conceptual Question.** Fill in the blanks: If A is an $n \times m$ matrix, $\text{Col } A$ is a subspace of _____ and $\text{Null } A$ is a subspace of _____
8. **Conceptual Question.** Is it possible that $\text{Col } A = \text{Null } A$? If so, give an example. If not, explain why not.
9. **Conceptual Question.** Suppose A is an $n \times m$ matrix and B is an $m \times k$ matrix such that $\text{Null } A = \text{Col } B$. What can you say about AB ?
10. **Conceptual Question.** Give an example of matrices A and B such that neither A nor B is invertible, but AB is invertible. Can A and B be square?
11. **Conceptual Question.** If A is an $n \times m$ matrix then how many solutions does $A\mathbf{x} = \mathbf{b}$ have if:
- (a) $\text{Null } A = \{0\}$ and $\mathbf{b} \in \text{Col } A$?
- (b) $\text{Null } A \neq \{0\}$ and $\mathbf{b} \in \text{Col } A$?
- (c) $\mathbf{b} \notin \text{Col } A$?
12. **Conceptual Question.** If A is an $n \times n$ invertible matrix, what are $\text{Null } A$ and $\text{Col } A$?