Math 54 Midterm 2 Review

1. Which of the following are vector spaces? (For an extra challenge, for each set that is a vector space, try to figure out its dimension.)

   (a) The set of 5×5 matrices A such that $A^T = A$.

   (b) The set of linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^2$ that are onto.

   (c) The set of all polynomials with coefficients in the real numbers that have the form $ax^3 + bx$.

   (d) The set of convergent sequences of real numbers.

> (a) Vector space.
>
> (b) Not a vector space. For instance, it does not contain the linear transformation that maps all vectors to $\mathbf{0}$ (which is the zero vector in the vector space of all linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^2$).
>
> (c) Vector space.
>
> (d) Vector space.

2. Let V be the subspace of the vector space of continuous functions that is spanned by the set $\{\sin(x), \cos(x), e^{e^x}\}$ (you may assume without proof that these three functions are linearly independent). Do the functions f, g, h form a basis for V?

$$\begin{aligned} f(x) &= 3\sin(x) + 2\cos(x) + e^{e^x} \\ g(x) &= \sin(x) + 2\cos(x) \\ h(x) &= 5\sin(x) + 6\cos(x) + e^{e^x} \end{aligned}$$

> Note that the set $\mathcal{B} = \{\sin(x), \cos(x), e^{e^x}\}$ forms a basis for V. So we can write everything as a coordinate vector in this basis and use the fact that $\mathbb{R}^3$ is isomorphic to V.
>
> $$[f]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad [g]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad [h]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix}$$
>
> To check if these three vectors in $\mathbb{R}^3$ are a basis, we can write them as the columns of a matrix and check if it has a pivot in every row.
>
> $$\begin{bmatrix} 3 & 1 & 5 \\ 2 & 2 & 6 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$
>
> Since this matrix does not have a pivot in every row, $\{[f]_{\mathcal{B}}, [g]_{\mathcal{B}}, [h]_{\mathcal{B}}\}$ does not form a basis for $\mathbb{R}^3$ so $\{f, g, h\}$ does not form a basis for V.

3. Let $\mathcal{B} = \{1 + x, x + x^2, 1 + x^2\}, \mathcal{C} = \{1 + x + x^2, 2 + x + x^2, 3x^2\}$ and $T\colon \mathbb{P}_2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(p) = \begin{bmatrix} p(2) \\ p(1) + p(3) \end{bmatrix}.$$

(a) Find the matrix for T relative to $\mathcal{B}$ and the standard basis of $\mathbb{R}^2$.

(b) Find the change of basis matrix from $\mathcal{C}$ to $\mathcal{B}$.

(c) Use your answers to parts (a) and (b) to find the matrix for T relative to $\mathcal{C}$ and the standard basis for $\mathbb{R}^2$.

(a)

$$_{\mathcal{E}}[T]_{\mathcal{B}} = [[T(1+x)]_{\mathcal{E}}\ [T(x+x^2)]_{\mathcal{E}}\ [T(1+x^2)]_{\mathcal{E}}] = \begin{bmatrix} 3 & 6 & 5 \\ 6 & 14 & 12 \end{bmatrix}$$

(b) To solve this problem, we will write everything as a coordinate vector in the basis $\mathcal{D} = \{1, x, x^2\}$ and then solve the problem in $\mathbb{R}^3$ in the usual way. If we do this, we simply need to find the change of basis matrix from the basis of $\mathbb{R}^3$ consisting of the columns of matrix B (where the columns of B are the vectors in $\mathcal{C}$ written as coordinate vectors in basis $\mathcal{D}$) to the basis of $\mathbb{R}^3$ consisting of the columns of matrix A (where the columns of A are the vectors in $\mathcal{B}$ written as coordinate vectors in basis $\mathcal{D}$).

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The change of basis matrix is just $A^{-1}B$, and one way to find it is to row reduce $[A \mid B]$ to $\left[I \mid A^{-1}B\right]$. This gives us the matrix

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 1/2 & 1 & -3/2 \\ 1/2 & 0 & 3/2 \\ 1/2 & 1 & 3/2 \end{bmatrix}$$

(c)

$$_{\mathcal{E}}[T]_{\mathcal{B}} = {}_{\mathcal{E}}[T]_{\mathcal{B}} \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 7 & 8 & 12 \\ 16 & 18 & 30 \end{bmatrix}$$

4. Let $\mathbf{v}$ and $\mathbf{u}$ be eigenvectors of a matrix A with different eigenvalues. Show that $\mathbf{u} + \mathbf{v}$ is not an eigenvector of A.

Let λ and μ be the eigenvalues corresponding to $\mathbf{v}$ and $\mathbf{u}$ respectively. Suppose for contradiction that $\mathbf{u}+\mathbf{v}$ is an eigenvector of A and let γ be the corresponding eigenvalue. Then we have

$$A(\mathbf{v}+\mathbf{u}) = A\mathbf{v} + A\mathbf{u} = \lambda\mathbf{v} + \mu\mathbf{u}$$

and

$$A(\mathbf{v}+\mathbf{u}) = \gamma(\mathbf{v}+\mathbf{u}) = \gamma\mathbf{v} + \gamma\mathbf{u}.$$

Subtracting these two equations gives us

$$\mathbf{0} = (\lambda-\gamma)\mathbf{v} + (\mu-\gamma)\mathbf{u}.$$

Since $\mathbf{v}$ and $\mathbf{u}$ are eigenvectors for distinct eigenvalues, then as shown in class they are linearly independent. Thus we must have $\lambda-\gamma=0$ and $\mu-\gamma=0$. But this implies that $\lambda=\gamma$ and $\mu=\gamma$, contradicting the assumption that $\lambda\neq\mu$.

5. Let $\mathbf{v}_1,\ldots,\mathbf{v}_k$ be eigenvectors of a matrix A with distinct eigenvalues. Show that no linear combination of $\mathbf{v}_1,\ldots,\mathbf{v}_k$ is an eigenvector of A.

This problem is not quite correct as stated: $\mathbf{v}_1$ is a linear combination of $\mathbf{v}_1,\ldots,\mathbf{v}_k$ and it is an eigenvector of A. The problem should have stated that a linear combination of $\mathbf{v}_1,\ldots,\mathbf{v}_k$ can only be an eigenvector of A if it is simply a multiple of one of the $\mathbf{v}_i$ (i.e. all but one of the coefficients are 0).

Let $a_1,\ldots,a_n$ be scalars such that at least two of the a_i are nonzero. Without loss of generality, we assume that a_1 and a_2 are nonzero. Let $\mathbf{x} = a_1\mathbf{v}_1 + \ldots + a_k\mathbf{v}_k$. Suppose for contradiction that $\mathbf{x}$ is an eigenvector of A, with eigenvalue γ. Then we have

$$A\mathbf{x} = \gamma\mathbf{x} = \gamma(a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n) = \gamma a_1\mathbf{v}_1 + \ldots + \gamma a_n\mathbf{v}_n$$

And on the other hand we have

$$A\mathbf{x} = A(a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n) = \lambda_1 a_1\mathbf{v}_1 + \ldots + \lambda_n a_n\mathbf{v}_n$$

Subtracting the second equation from the first gives us

$$\mathbf{0} = (\gamma-\lambda_1)a_1\mathbf{v}_1 + \ldots + (\gamma-\lambda_n)a_n\mathbf{v}_n$$

Since $\mathbf{v}_1,\ldots,\mathbf{v}_n$ are eigenvectors of A with distinct eigenvalues, they are linearly independent. Thus the above equation implies that for each i, $(\gamma-\lambda_i)a_i = 0$. And if $a_i\neq 0$ this implies that $\gamma-\lambda_i=0$ and thus that $\gamma=\lambda_i$. Since a_1 and a_2 are both nonzero by assumption, we have $\lambda_1=\gamma=\lambda_2$, contradicting the assumption that λ_1 and λ_2 are distinct.

6. Suppose A is a 5×5 matrix whose characteristic polynomial is $\lambda^3(\lambda-1)(\lambda-2)$. What are the possible values for rank A? For which of these values is A diagonalizable?

Since the multiplicity of 0 as an eigenvalue of A is 3, the eigenspace $\text{Null}(A - 0I) = \text{Null}\, A$ has dimension at least 1 and at most 3. By the rank theorem, this implies that $\text{rank}\, A$ is between $5 - 3 = 2$ and $5 - 1 = 4$. Since A is diagonalizable if and only if the dimension of its eigenspaces adds up to 5, it is diagonalizable if and only if $\text{Null}\, A$ has dimension 3. Thus A is diagonalizable if and only if $\text{rank}\, A$ is 2.

7. Find a formula for A^n where

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}.$$

Diagonalize A. To compute the eigenvalues we can find the characteristic polynomial or note that it is upper triangular, so the eigenvalues are the entries along the diagonal– 2 and 3. Now we need to find an eigenvector basis.

$$E_2 = \text{Null}(A - 2I) : A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$E_3 = \text{Null}(A - 3I) : A - 3I = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

So two linearly independent eigenvectors (with eigenvalues 2 and 3 respectively) are

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus we can write

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}$$

And therefore we have

$$\begin{aligned} A^n &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2^n & 3^n - 2^n \\ 0 & 3^n \end{bmatrix} \end{aligned}$$

8. Mark each of the following statements true or false. For each statement, either give a proof that it is always true or give a counterexample to show it can be false.

   (a) Let A be an $n \times n$ matrix with only positive eigenvalues such that there is an orthogonal basis for $\mathbb{R}^n$ consisting of eigenvectors of A. If $\mathbf{x}$ is a nonzero vector then $\mathbf{x} \cdot (A\mathbf{x})$ is positive.

(b) Every diagonalizable matrix is invertible.

(c) Every invertible matrix is diagonalizable.

(d) Every matrix with a repeated eigenvalue is not diagonalizable.

(e) Let V be a subspace of $\mathbb{R}^n$ and W a subspace of V. Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$, $\mathbf{y}$ the projection of $\mathbf{x}$ on the subspace V and $\mathbf{z}$ the projection of $\mathbf{y}$ on the subspace W. Then $\mathbf{z}$ is the projection of $\mathbf{x}$ on W.

(f) If $\mathbf{v}$ and $\mathbf{u}$ are eigenvectors of a matrix A with distinct eigenvalues then $\mathbf{v}$ and $\mathbf{u}$ are orthogonal.

Note: on an exam you should actually prove that your counterexamples are in fact valid counterexamples.

(a) True. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be an orthogonal basis of $\mathbb{R}^n$ consisting of eigenvectors of A with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $\mathbf{x}$ be any nonzero vector. Since the $\mathbf{v}_i$'s form a basis, there exist scalars $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n$. We have

$$\begin{aligned}\mathbf{x} \cdot (A\mathbf{x}) &= (c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n) \cdot (A(c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n)) \\ &= (c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n) \cdot (\lambda_1 c_1\mathbf{v}_1 + \ldots + \lambda_n c_n\mathbf{v}_n)\end{aligned}$$

By linearity of the dot product in each coordinate and by orthogonality of the $\mathbf{v}_i$'s, this is equal to

$$= \lambda_1 c_1^2(\mathbf{v}_1 \cdot \mathbf{v}_1) + \ldots + \lambda_n c_n^2(\mathbf{v}_n \cdot \mathbf{v}_n)$$

which is always nonnegative, and positive as long as one of the c_i's is nonzero (which must be the case if $\mathbf{x}$ is nonzero).

(b) False. Counterexample: $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$.

(c) False. Counterexample: $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$.

(d) False. Counterexample: $\left[\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right]$.

(e) True. Let $\mathbf{y}' = \mathbf{x} - \mathbf{y}$ and let $\mathbf{z}' = \mathbf{y} - \mathbf{z}$. So as we have seen in class, $\mathbf{y}' \in V^\perp$ and $\mathbf{z}' \in W^\perp$. Since $W \subseteq V$, $\mathbf{y}'$ is also perpindicular to everything in W– i.e. $\mathbf{y}' \in W^\perp$. By a homework exercise, $W^\perp$ is a subspace and so $\mathbf{y}' + \mathbf{z}' \in W^\perp$. So we have

$$\mathbf{x} = \mathbf{z} + (\mathbf{z}' + \mathbf{y}')$$

where $\mathbf{z} \in W$ and $\mathbf{z}' + \mathbf{y}' \in W^\perp$. Since the decomposition of $\mathbf{x}$ into an element of W and an element of $W^\perp$ is unique, we must have $\mathbf{z} = \text{Proj}_W(\mathbf{x})$.

(f) False. Take any two vectors that are linearly independent and not orthogonal. Define a linear transformation for which both vectors are eigenvectors. As an example, take the two vectors to be $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ and let $P = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. Then $A = P\left[\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right]P^{-1}$ has the two chosen vectors as eigenvectors.