Math 54 Midterm 1 Review Solutions

1. Suppose the following system has exactly two free variables: x_3 and x_4.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + a_{15}x_5$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + a_{25}x_5$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 + a_{35}x_5$$

(a) How many solutions does the homogeneous equation have?

A: Since there are free variables, the homogeneous equation has infinitely many solutions.

(b) Let $\mathbf{v}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}, \ldots, \mathbf{v}_5 = \begin{bmatrix} a_{15} \\ a_{25} \\ a_{35} \end{bmatrix}$. Are $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ linearly independent?

A: These vectors are just the columns of the coefficient matrix from the system written above. Since the system has free variables, the coefficient matrix does not have a pivot in every column. So the vectors are not linearly independent.

(c) Let $A = [\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3\mathbf{v}_4\mathbf{v}_5]$. Find a basis for Col A.

A: One basis for the column space of a matrix is simply the columns of the matrix that have pivot positions. So in this case one basis for Col A is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5\}$.

(d) What is rank A?

A: The rank of a matrix is the dimension of its column space– i.e. the number of vectors in a basis for the column space. From part (c), we can see that a basis for Col A has 3 vectors, so the rank of A is 3.

(e) Do $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ span $\mathbb{R}^3$?

A: This is equivalent to asking if every row of A has a pivot. Since there are three pivot columns and three rows, there must be a pivot in every row. So these vectors do span $\mathbb{R}^3$.

(f) Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. Is T one-to-one? Is T onto?

A: T is one-to-one if and only if its standard matrix has a pivot in every column. Since T's standard matrix is A, this means that T is not one-to-one (because we have already said that A does not have a pivot in every column).

On the other hand, T is onto if and only if its standard matrix has a pivot in every row. As we have already seen (see part (e)), A does have a pivot in every row, so T is onto.

(g) Does the matrix equation $A\mathbf{x} = \mathbf{b}$ have a solution for every $\mathbf{b} \in \mathbb{R}^3$? When it does have a solution, is the solution unique?

A: A solution exists for all $\mathbf{b} \in \mathbb{R}^3$ if and only if A has a pivot in every row. So a solution does always exist. A solution to $A\mathbf{x} = \mathbf{b}$ is only unique if A has a pivot in every column. So in this case the solution is not unique.

2. Let A be an $n \times n$ matrix.

(a) Simplify $(I + A + A^2 + \ldots + A^{m-1})(I - A)$

A: Since we can distribute matrix multiplication we get

$$\begin{aligned}(I + A + A^2 + \ldots + A^{m-1})(I - A) &= I(I - A) + A(I - A) + A^2(I - A) + \ldots + A^{m-1}(I - A)\\ &= (I - A) + (A - A^2) + (A^2 - A^3) + \ldots + (A^{m-1} - A^m)\\ &= I + (A - A) + (A^2 - A^2) + \ldots + (A^{m-1} - A^{m-1}) + A^m\\ &= I + A^m\end{aligned}$$

(b) If $(I - A)$ is invertible, find an expression equivalent to $(I - A^m)(I - A)^{-1}$ (hint: use part (a)).

A: By part (a), $(I + A + A^2 + \ldots + A^{m-1})(I - A) = (I - A^m)$. Multiplying both sides by the inverse of $(I - A)$ we get $I + A + A^2 + \ldots + A^{m-1} = (I - A^m)(I - A)^{-1}$.

3. Find a basis for Col A and a basis for Null A.

$$A = \begin{bmatrix} 0 & 2 & 2 & -2\\ 1 & -1 & 0 & 3\\ 2 & 1 & 3 & 3\\ 3 & -1 & 2 & 7\end{bmatrix}$$

A: To find a basis for Col A we row reduce A to find which columns have pivots.

$$\begin{bmatrix} 0 & 2 & 2 & -2\\ 1 & -1 & 0 & 3\\ 2 & 1 & 3 & 3\\ 3 & -1 & 2 & 7\end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 0 & 3\\ 0 & 2 & 2 & -2\\ 2 & 1 & 3 & 3\\ 3 & -1 & 2 & 7\end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 0 & 3\\ 0 & 2 & 2 & -2\\ 0 & 3 & 3 & -3\\ 0 & 2 & 2 & -2\end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & -1 & 0 & 3\\ 0 & 1 & 1 & -1\\ 0 & 3 & 3 & -3\\ 0 & 2 & 2 & -2\end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 0 & 3\\ 0 & 1 & 1 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

At this point we can see that the first and second columns are the only pivot columns so one basis for Col A is the first and second columns of A:

$$\left\{ \begin{bmatrix} 0\\ 1\\ 2\\ 3\end{bmatrix}, \begin{bmatrix} 2\\ -1\\ 1\\ -1\end{bmatrix} \right\}$$

A few words of caution here: applying elementary row operations to a matrix *changes the column space* so we had to take the columns from A itself to get a basis for Col A– the pivot columns of A transformed into RREF would not work. The only purpose of row reducing A here was to figure out which columns of A had pivot positions. Also, this is just one basis for Col A– there are many others.

To find a basis for Null A we need to find solutions to the homogeneous equation. So we need to row reduce, which we have fortunately already done above. Putting the matrix all the way into RREF, we get

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the solution set for the homogeneous equation consists of all vectors of the form

$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} x_4$$

So a basis for Null A is

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

4. True or False: If A is an $n \times m$ matrix and B is an $m \times p$ matrix such that Col $B =$ Null A, then $AB = 0$.

   A: True. Let $\mathbf{x}$ be any vector in $\mathbf{R}^p$. By the definition of multiplying a matrix by a vector, $B\mathbf{x} \in$ Col B. Since we have assumed Col $B =$ Null A, this means $B\mathbf{x} \in$ Null A. So by definition of Null A, $A(B\mathbf{x}) = \mathbf{0}$. Thus $(AB)\mathbf{x} = \mathbf{0}$. Since AB sends all vectors to $\mathbf{0}$, AB is 0.

5. True or False: If A is a 2×10 matrix then $\dim \text{Null } A \geq 8$.

   A: True. Since A only has 2 rows, it has at most 2 pivot positions. Since the rank of A is equal to the number of pivots, this means that the rank of A is at most 2. But the rank of A and the dimension of the null space have to sum up to the number of columns, which is 10. So $\dim \text{Null } A \geq 8$.

6. True or False: If $\mathbf{v}_1, \ldots, \mathbf{v_m}$ are a set of vectors that span $\mathbb{R}^n$ and T and S are linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^p$ such that $T(\mathbf{v}_i) = S(\mathbf{v}_i)$ for all $i \leq m$ then $S = T$.

   A: True. Let $\mathbf{x}$ be any vector in $\mathbf{R}^n$. Since $\mathbf{v}_1, \ldots, \mathbf{v_m}$ span $\mathbb{R}^n$, $\mathbf{x}$ can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v_m}$. Thus there exist scalars $a_1, \ldots, a_m$ such that $\mathbf{x} = a_1\mathbf{v}_1 + \ldots + a_m\mathbf{v_m}$. So we have

$$\begin{aligned} T(\mathbf{x}) &= T(a_1\mathbf{v}_1 + \ldots + a_m\mathbf{v_m}) \\ &= a_1T(\mathbf{v}_1) + \ldots + a_mT(\mathbf{v_m}) && \text{since } T \text{ is a linear transformation} \\ &= a_1S(\mathbf{v}_1) + \ldots + a_mS(\mathbf{v_m}) && \text{by the assumption in the problem statement} \\ &= S(a_1\mathbf{v}_1 + \ldots + a_m\mathbf{v_m}) && \text{since } S \text{ is a linear transformation} \\ &= S(\mathbf{x}) \end{aligned}$$

   Since T and S agree on all vectors in the domain, by definition $T = S$.