Math 54 Midterm 1 Review

1. Suppose the following system has exactly two free variables: x_3 and x_4.

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + a_{15}x_5 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + a_{25}x_5 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 + a_{35}x_5
\end{aligned}$$

   (a) How many solutions does the homogeneous equation have?

   (b) Let $\mathbf{v}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}, \ldots, \mathbf{v}_5 = \begin{bmatrix} a_{15} \\ a_{25} \\ a_{35} \end{bmatrix}$. Are $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ linearly independent?

   (c) Let $A = [\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3 \mathbf{v}_4 \mathbf{v}_5]$. Find a basis for Col A.

   (d) What is rank A?

   (e) Do $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ span $\mathbb{R}^3$?

   (f) Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. Is T one-to-one? Is T onto?

   (g) Does the matrix equation $A\mathbf{x} = \mathbf{b}$ have a solution for every $\mathbf{b} \in \mathbb{R}^3$? When it does have a solution, is the solution unique?

2. Let A be an $n \times n$ matrix.

   (a) Simplify $(I + A + A^2 + \ldots + A^{m-1})(I - A)$

   (b) If $(I - A)$ is invertible, find an expression equivalent to $(I - A^m)(I - A)^{-1}$ (hint: use part (a)).

3. Find a basis for Col A and a basis for Null A.

$$A = \begin{bmatrix} 0 & 2 & 2 & -2 \\ 1 & -1 & 0 & 3 \\ 2 & 1 & 3 & 3 \\ 3 & -1 & 2 & 7 \end{bmatrix}$$

4. True or False: If A is an $n \times m$ matrix and B is an $m \times p$ matrix such that Col $B =$ Null A, then $AB = 0$.

5. True or False: If A is a 2×10 matrix then $\dim \text{Null } A \geq 8$.

6. True or False: If $\mathbf{v}_1, \ldots, \mathbf{v_m}$ are a set of vectors that span $\mathbb{R}^n$ and T and S are linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^p$ such that $T(\mathbf{v}_i) = S(\mathbf{v}_i)$ for all $i \leq m$ then $S = T$.