MATH 54, FALL 2016, QUIZ 12 SOLUTIONS

In each of the following problems, find a solution to the heat equation, $\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$, with boundary values $u(0,t) = u(L,t) = 0$ and initial conditions as given in each problem.

(1) Find a solution to the heat equation with initial condition $u(x,0) = 3f(x)$ where $f\colon [0,L] \to \mathbb{R}$ is a continuous function such that $f(0) = f(L) = 0$ and

$$\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx = \frac{1}{n^2}.$$

Recall that for any sufficiently "nice" function g defined on the interval $[0,L]$ (for instance, any function such that $g(0) = g(L) = 0$ and g' is piecewise continuous) the solution to the heat equation with the given boundary values and initital conditions given by g is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{\langle g, \sin\left(\frac{n\pi x}{L}\right)\rangle}{\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\rangle} e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

where

$$\langle g, \sin\left(\frac{n\pi x}{L}\right)\rangle = \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) \, dx$$

$$\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\rangle = \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) \, dx.$$

Using the values given in the question and the hint, we have

$$\langle 3f, \sin\left(\frac{n\pi x}{L}\right)\rangle = \frac{3}{n^2}$$

$$\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\rangle = \frac{L}{2}.$$

So the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{6}{Ln^2} e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right).$$

(2) Find a solution to the heat equation with initial condition

$$u(x,0) = -5 \sin\left(\frac{32\pi x}{L}\right) + 13 \sin\left(\frac{307\pi x}{L}\right).$$

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We could go through the same process as in part (a)–i.e. find the Fourier coefficients for the given function–but in this case it's not necessary since the initial condition is already written as a linear combination of sine functions. So the solution is

$$u(x,t) = -5e^{-\beta\left(\frac{32\pi}{L}\right)^2 t}\sin\left(\frac{32\pi x}{L}\right) + 13e^{-\beta\left(\frac{307\pi}{L}\right)^2 t}\sin\left(\frac{307\pi x}{L}\right).$$

Hint: $\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)\,dx = \frac{L}{2}$.