Weihrauch reducibility and the Solecki dichotomy

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These notes are concerned with the following question due to Carroy.

Question 0.1. What is the structure of $\leq_{sW}^c$ on the Borel functions on ω^ω? In particular, is it a well-founded quasi order?

The answer to this question is still open. However, there are several variations on $\leq_{sW}^c$ for which we can give a complete answer. In particular, we can answer it in each of the following cases.

- $\leq_{sW}^c$ restricted to 0/1-valued functions.
- Parallelized Weihrauch reducibility, which we will denote $\leq_{sW}^{c,p}$. Recall that $f \leq_{sW}^{c,p} g$ means $\hat{f} \leq_{sW}^c \hat{g}$.
- Non-uniform Weihrauch reducibility. We will denote the strong version by $\leq_{sW}^{c,nu}$ and the weak version by $\leq_{W}^{c,nu}$.

The first of these reducibilities essentially corresponds to the Wadge degrees (except that the degree of a set and its complement are conflated) and its classification is well-known. The classification of Borel functions under parallelized Weihrauch reducibility is due to Day, Downey and Westrick in [**?**] and the classification under non-uniform Weihrauch reducibility follows from a generalization by Marks and Montalban [**?**] of a theorem due to Solecki [**?**]. The main goal of these notes is to explain these classifications.

1 Weihrauch Reducibility on $0/1$-Valued Functions

2 Parallelized Weihrauch Reducibility

In this section, we will analyze the structure of $\leq_{sW}^{c,p}$ on the Borel functions. In particular, we will show that every Borel function is equivalent to iterate of the Turing jump.

Definition 2.1. For Borel functions f and g, let $G(f, g)$ denote the following game with moves in ω.

3 Non-uniform Weihrauch Reducibility

First recall the definition of non-uniform Weihrauch reducibility.

Definition 3.1. For any functions $f, g \colon \omega^\omega \to \omega^\omega$, $f \leq_{sW}^{c,nu} g$ means that there is a countable partition $\langle A_n \rangle_{n\in\omega}$ of ω^ω such that for each n, $f \restriction_{A_n} \leq_{sW}^c g$.

Another way to think about this is that we can reduce f to g using finite advice: i.e. for every x some magical oracle gives us a finite number of bits (e.g. telling us which of the A_n's contains x) which helps us choose which Weihrauch reduction to use.

It turns out that under this notion of reducibility, almost every function is equivalent to J_α for some $\alpha < \omega_1$. The only wrinkle is that for every limit α, there is one extra equivalence class that is strictly below J_α but above J_β for all $\beta < \alpha$. This classification follows easily from an unpublished theorem due to Marks and Montalban [**?**] which in turn is a generalization of a theorem due to Solecki.

Theorem 3.2 (Solecki dichotomy)**.** For every Borel function $f \colon \omega^\omega \to \omega^\omega$, either $f \leq_{sW}^{c,nu} \mathsf{Id}$ or $\mathsf{J} \leq_{sW}^c f$.

Theorem 3.3 (Generalized Solecki dichotomy)**.** Suppose f is a Borel function and $\alpha < \omega_1$. Then one of the following must hold:

(1) There is some countable partition $\langle A_n \rangle_{n\in\omega}$ of ω^ω and some countable sequence $\langle \beta_n \rangle_{n\in\omega}$ of ordinals below α such that for all n, $f \restriction_{A_n} \leq_{sW}^c J_{\beta_n}$.

(2) $J_\alpha \leq_{sW}^c f$.

Comment. Note that the Solecki dichotomy implies that there are no functions which are strictly in-between Id and J for $\leq_{sW}^{c,nu}$. However, it actually gives us more than we need to prove that; it tells us not just that $f \not\leq_{sW}^{c,nu} \mathsf{Id}$ implies $J \leq_{sW}^{c,nu} f$ but also that $J \leq_{sW}^{c} f$. We will not need this extra strength, but it is interesting to note.

Comment. The generalized Solecki dichotomy looks a bit more complicated than you might expect. In particular, you might expect it to say something like the following.

> For every Borel function f and $\alpha < \omega_1$, either there is some $\beta < \alpha$ such that $f \leq_{sW}^{c,nu} J_\beta$ or $J_\alpha \leq_{sW}^{c} f$.

However, this statement is false. To see why, consider the following function

$$f(x) = J_n(x) \text{ where } n \text{ is the first digit of } x.$$

It is easy to see that this function is not reducible to any J_n for $n \in \omega$ but also that J_ω does not reduce to it.

Comment. Since (in our notation) $J_1 = \mathsf{Id}$ and $J_2 = J$, the generalized Solecki dichotomy includes the Solecki dichotomy as a special case.

Theorem 3.4. For every Borel function $f\colon \omega^\omega \to \omega^\omega$ there is some $\alpha < \omega_1$ such that one of the following holds.

(1) $f \equiv_{sW}^{c,nu} J_\alpha$

(2) For all $\beta < \alpha$, $J_\beta \leq_{sW}^{c} f$ and there is some countable partition $\langle A_n \rangle_{n\in\omega}$ of ω^ω and some countable sequence $\langle \beta_n \rangle_{n\in\omega}$ of ordinals below α such that for all n, $f \restriction_{A_n} \leq_{sW}^{c} J_{\beta_n}$.

Furthermore, if f and g fall into case 2 above for the same α then $f \equiv_{sW}^{c,nu} g$.

Proof. Let α be the least ordinal such that $f \leq_{sW}^{c,nu} J_\alpha$. First suppose that there is no countable parition of ω^ω and sequence of ordinals below α which satisfy the assumptions of the generalized Solecki dichotomy. Then by that theorem, we have that $J_\alpha \leq_{sW}^{c} f$ and hence $f \equiv_{sW}^{c,nu} J_\alpha$ and so we are done.

Now suppose we do have such a partition $\langle A_n \rangle_{n\in\omega}$ and sequence $\langle \beta_n \rangle_{n\in\omega}$. We have two things to prove: that α is a limit and that for all $\beta < \alpha$, $J_\beta \leq_{sW}^{c} f$.

First let's show that α is a limit ordinal. If not then $\alpha = \gamma + 1$ for some γ. Therefore for each n, $\beta_n \leq \gamma$ and thus we have $f \restriction_{A_n} \leq_{sW}^{c} J_{\beta_n} \leq_{sW}^{c} J_\gamma$. This shows that $f \leq_{sW}^{c,nu} J_\gamma$, which contradicts our choice of α.

Now let $\beta < \alpha$ and let's show that $J_\beta \leq_{sW}^{c} f$. By choice of α, we have that $f \not\leq_{sW}^{c,nu} J_\beta$. It is straightforward to check that the generalized Solecki dichotomy applies and so we have $J_\beta \leq_{sW}^{c} f$ as desired.

Now we want to verify the last sentence of the theorem. Suppose $\alpha < \omega_1$ is a limit ordinal and that f and g are two Borel functions such that there exist partitions $\langle A_n \rangle_{n\in\omega}$ and $\langle B_n \rangle_{n\in\omega}$ and sequence $\langle \beta_n \rangle_{n\in\omega}$ and $\langle \gamma_n \rangle_{n\in\omega}$ as in the theorem statement. We want to show $f \equiv_{sW}^{c,nu} g$. By symmetry it is enough to show that $f \leq_{sW}^{c,nu} g$. Note that for each n we have $\beta_n < \alpha$ and thus $J_{\beta_n} \leq_{sW}^{c} g$ and thus we get

$$f \restriction_{A_n} \leq_{sW}^{c} J_{\beta_n} \leq_{sW}^{c} g.$$

Hence the partition $\langle A_n \rangle_{n\in\omega}$ witnesses that $f \leq_{sW}^{c,nu} g$.

4 The Solecki Dichotomy and the Posner-Robinson Theorem

5 Silver's Theorem and the Baby Solecki Dichotomy

6 The Solecki Dichotomy

7 Carroy's Question

8 Weihrauch Reducibility and Martin's Conjecture