Math 10B, Quiz 12 Solutions

1. (9 points) Each part below is worth 3 points. Even if you get the first two parts wrong, you can receive credit for part (c) as long as your answer is consistent with your answers to parts (a) and (b).

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

(a) Find the eigenvalues of A.

Solution: First we need to find the characteristic polynomial of A.

$$\begin{aligned} \det(A - \lambda I) &= \det\left(\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\ &= \det \begin{bmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{bmatrix} \\ &= (1-\lambda)(2-\lambda) - 4 \cdot 3 \\ &= 2 - 3\lambda + \lambda^2 - 12 \\ &= \lambda^2 - 3\lambda - 10 \\ &= (\lambda - 5)(\lambda + 2) \end{aligned}$$

The eigenvalues of A are the roots of the characteristic polynomial, which in this case are $\boxed{5 \text{ and } -2}$.

(b) Find all of the eigenvectors of A.

Solution: Eigenvectors for the eigenvalue 5: An eigenvector for A with eigenvalue 5 is just a non-zero solution to $Av = 5v$, or equivalently, $(A - 5I)v = 0$ (where 0 here denotes the all-zeros vector). We can find such solutions using Gaussian elimination:

$$\left[\begin{array}{cc|c} 1-5 & 4 & 0 \\ 3 & 2-5 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -4 & 4 & 0 \\ 3 & -3 & 0 \end{array}\right] \xrightarrow{-\frac{1}{4}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right] \xrightarrow{R_2 - 3R_1 \to R_2} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So any solution to

$$(A - 5I)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Must have $x_1 - x_2 = 0$ and hence $x_1 = x_2$. So the eigenvectors of A with eigenvalue 5 are all vectors of the form

$$\begin{bmatrix} a \\ a \end{bmatrix}$$

where a is any nonzero real number.

Eigenvectors for the eigenvalue -2:

$$\left[\begin{array}{cc|c} 1-(-2) & 4 & 0 \\ 3 & 2-(-2) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 3 & 4 & 0 \\ 3 & 4 & 0 \end{array}\right] \xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{cc|c} 3 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So any solution to

$$(A - (-2)I)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

must have $3x_1 + 4x_2 = 0$. So $x_1 = -\frac{4}{3}x_2$ and the eigenvectors of A with eigenvalue -2 are all vectors of the form

$$\begin{bmatrix} -\frac{4}{3}b b \end{bmatrix}$$

where b is any nonzero real number.

Common Mistakes: Many people just found one eigenvector each for the two eigenvalues, but the problem said to find *all* eigenvectors.

(c) Find all solutions to the following system of differential equations.

$$\begin{aligned} x' &= x + 4y \\ y' &= 3x + 2y \end{aligned}$$

Solution: Note that the coefficient matrix here is just the matrix A from above. So using the solutions to parts (a) and (b), the solutions here are

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} -4/3 \\ 1 \end{bmatrix} = \begin{bmatrix} C_1 e^{5t} - \frac{4}{3} C_2 e^{-2t} \\ C_1 e^{5t} + C_2 e^{-2t} \end{bmatrix}$$

where C_1 and C_2 are any real numbers.

Comment: Note that here, C_1 and C_2 *are* allowed to be zero.

2. (2 points) v is an eigenvector of B.

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 5 \\ 0 & 0 & 2 & 3 & 0 \\ 6 & -3 & -3 & 0 & 5 \\ 2 & 2 & 2 & -2 & 1 \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

√ **True** ○ False

Solution: To check if a vector is an eigenvector of a matrix, we just need to multiply the matrix and the vector and check if the result is a multiple of the original vector. In this case, we get

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 5 \\ 0 & 0 & 2 & 3 & 0 \\ 6 & -3 & -3 & 0 & 5 \\ 2 & 2 & 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \\ 5 \end{bmatrix} = 5 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

3. (2 points) It makes sense to ask for the eigenvalues of the following matrix

$$\begin{bmatrix} 2 & 4 & 6 & 1 \\ -2 & -2 & -1 & 0 \\ 3 & 12 & 16 & 0 \end{bmatrix}$$

○ True √ **False**

Solution: It only makes sense to talk about eigenvectors of square matrices. Why? Well, remember that an eigenvector of a matrix is a vector that when multiplied by the matrix, results in a multiple of the original vector. But if the matrix is $n \times m$ then we can only multiply it by length m vectors and the result is a length n vector. And one vector can only be a multiple of another if they have the same length, which can only happen if $n = m$.

Comment: For non-square matrices, there are concepts that are analogous to eigenvalues and eigenvectors called singular values and right and left singular vectors. And these are very important in many applications of linear algebra (including, but by no means limited to, statistics and machine learning). But understanding singular values requires more linear algebra than we have time to learn in this class.

4. (2 points) A student is asked to find the eigenvalues and eigenvectors of some matrix C. The student believes that one of the eigenvalues is 3 and finds that $C - 3I$ is

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

The student then claims that the corresponding eigenvector is

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

because the only solution to $(C - 3I)v = 0$ is $x_1 = 0$ and $x_2 = 0$. The student

- √ **Found the wrong eigenvalue.**
- ○ May have found the correct eigenvalue but did not find a valid eigenvector.
- ○ Is completely correct.

Solution: If 3 really was an eigenvalue then any eigenvector with eigenvalue 3 would be a solution to $(C - 3I)v = 0$. Since the only solution to $(C - 3I)v = 0$ here is the all zeros vector, 3 must not be an eigenvalue of C and so the student made a mistake when finding eigenvalues.