

Combinatorics Worksheet 5: Combinations and Combinatorial Proofs

1. How many anagrams does “obfuscated” have?

Solution: Since all of the letters of “obfuscated” are distinct, this is just the number of permutations of a set with 10 elements, which is $\boxed{10!}$.

2. (a) How many anagrams does “banana” have?

Solution: This is similar to some of the problems from lecture. To find the answer, we will first solve a simpler problem. Suppose that instead of anagrams of “banana” we wanted to find the number of anagrams of “ $ba_1n_1a_2n_2a_3$ ” where all the letters are distinguishable. This is just the number of permutations of a set of 6 elements, which is $6!$.

In the actual problem however, some of the letters *are* indistinguishable—the three a’s, for instance—so $6!$ is not correct. In particular, for each anagram of “banana” there $3!2!$ corresponding anagrams of “ $ba_1n_1a_2n_1a_3$ ” because we can arrange the three a’s in any order and the two n’s in any order. As an example, the anagram “abnaan” corresponds to the 12 anagrams of “ $ba_1n_1a_2n_1a_3$ ” shown below:

- | | | |
|-----------------------|-----------------------|------------------------|
| 1. $a_1bn_1a_2a_3n_2$ | 5. $a_2bn_1a_1a_3n_2$ | 9. $a_3bn_1a_1a_2n_2$ |
| 2. $a_1bn_2a_2a_3n_1$ | 6. $a_2bn_2a_1a_3n_1$ | 10. $a_3bn_2a_1a_2n_1$ |
| 3. $a_1bn_1a_3a_2n_2$ | 7. $a_2bn_1a_3a_1n_2$ | 11. $a_3bn_1a_2a_1n_2$ |
| 4. $a_1bn_2a_3a_2n_1$ | 8. $a_2bn_2a_3a_1n_1$ | 12. $a_3bn_2a_2a_1n_1$ |

Since there are $3!2!$ times as many anagrams of $ba_1n_1a_2n_1a_3$ as there are anagrams of “banana” there are $\boxed{\frac{6!}{2!3!}}$ anagrams of “banana.”

Another method to solve this problem is to observe that to form an anagram of “banana” we can start with 6 empty spaces and first choose which 3 of those 6 spaces will be a’s, which 2 of the remaining 3 spaces will be n’s and which 1 of the remaining 1 spaces will be b. This gives an answer of $\boxed{\binom{6}{3}\binom{3}{2}\binom{1}{1}}$, sometimes also written as the multinomial coefficient $\binom{6}{3,2,1}$.

- (b) How many anagrams does “banana” have in which the three ‘a’s are next to each other?

Solution: We will treat the three a’s as a single letter, since they all have to appear together—e.g. this problem is equivalent to finding the number of anagrams of “bnnx” (where you can think of the “x” as representing the three a’s). By the same reasoning used in part (a) or in some of the problems in lecture, there are

$\boxed{\frac{4!}{2!} = \binom{4}{2}\binom{2}{1}\binom{1}{1} = \binom{4}{2,1,1}}$ such anagrams.

- (c) How many anagrams does “banana” have in which the three ‘a’s are next to each other and the two ‘n’s are *not* next to each other?

Solution: There are few enough letters involved that it is not hard to simply list all the possibilities. But if the word was longer, that strategy would no longer be feasible, so I will describe a solution that generalizes more easily.

We will first count the number of anagrams where the three a’s are next to each other and the two n’s are *also* next to each other. We can then subtract this from the answer from part (b) to find the answer to the original question.

To find the number of anagrams where the three a’s are next to each other and the two n’s are also next to each other, let’s use the same trick we used in part (b): consider the three a’s as a single letter and the two n’s as a single letter. This leaves us with just 3 letters, all of which are distinct. So there are $3!$ such anagrams.

So the answer to the original question is $\boxed{\frac{4!}{2!} - 3!}$.

Comment: Here’s a useful tip regarding this solution: we used a trick that many of you are by now familiar with, which was to count the size of the complement of the set we were interested in. But a subtle point about this, is that the complement of a set is always defined relative to a bigger set that contains it. In the solution above, that bigger set was the set of anagrams of “banana” in which the three a’s are next to each other. But we could have also tried using all anagrams of “banana” as the bigger set, or even all strings of six letters. However, these choices would have led to much more complicated solutions (do you see why?).

3. (a) Starting from a pool of n people, how many ways are there to select a committee of k people, one of whom is the president of the committee?

Solution: Either side of the equality in part (b) could be given as the answer. See part (b) for an explanation.

- (b) Use your answer to part (a) to prove that

$$k \binom{n}{k} = n \binom{n-1}{k-1}.$$

Solution: It is not difficult to give a proof of this fact just by doing a few algebraic manipulations. Instead, we will give a combinatorial proof—in other words we will show that both sides of the claimed equality can be given as answers to the same combinatorics question. In particular, we will show that both sides can be given as answers to part (a).

One solution to part (a) is as follows: To choose a committee of k members from a pool of n people, one of whom is the president of the committee, first choose k people from the pool of n and then choose one of those k people to be the president. There are $\binom{n}{k}$ ways to choose k people from n and k ways to choose one person from k . So there are $k \binom{n}{k}$ ways total.

Another solution to part (a) goes as follows: To choose a committee of k members from a pool of n people, one of whom is the president of the committee, first choose one out of the n people to be the president of the committee and then choose the other $k - 1$ committee members from the remaining $n - 1$ people. There are n ways to complete the first task and $\binom{n-1}{k-1}$ ways to complete the second. So there are $n\binom{n-1}{k-1}$ ways total.

Since $k\binom{n}{k}$ and $n\binom{n-1}{k-1}$ can both be given as answers to the same problem (i.e. both count the same thing), they must be equal.

4. Using any method you like (i.e. any chain of sound reasoning), prove that for any m, n , and k such that $k \leq n$ and $k \leq m$,

$$\binom{m+n}{k} = \sum_{i=0}^k \binom{m}{i} \binom{n}{k-i}.$$

Solution: Unlike problem 3(b), this equality is pretty tricky to prove with purely algebraic manipulations. So we will use a combinatorial proof instead. In particular, we will show that both sides of the equality count the number of ways to choose a single committee of k people from two pools of people, one of size n and one of size m .

One way to choose such a committee is to simply think of the two pools of people as one big pool of $n + m$ people and choose the k members of the committee from this big pool. There are $\binom{n+m}{k}$ ways to do that.

Another way to choose such a committee is to first decide how many people you will take from the pool of n people. Suppose you decide to take i people from the pool of n people. That means you need to choose $k - i$ people from the pool of m people. There are $\binom{n}{i}$ ways to choose i people from n and $\binom{m}{k-i}$ ways to choose $k - i$ people from m . So in the case where we pick i people from the pool of n people, there are $\binom{n}{i} \binom{m}{k-i}$ ways to choose the committee. Now we need to add up the number of ways to form the committee from each of the cases $i = 0, 1, \dots, k$, which gives us

$$\sum_{i=0}^k \binom{m}{i} \binom{n}{k-i}.$$