Matrix Algebra Worksheet 2 Solutions

1. Find the inverse of the following matrix

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & -5 \\ 0 & 0 & 6 \end{bmatrix}$$

Solution: We will use Gaussian elimination. The matrix is already upper triangular, but recall that for finding inverses, this is not enough.

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 4 & -5 & 0 & 1 & 0 \\ 0 & 0 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{6}R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 4 & -5 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/6 \end{array}\right]$$

$$\xrightarrow{R_2+5R_3 \to R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 5/6 \\ 0 & 0 & 1 & 0 & 0 & 1/6 \end{array}\right]$$

$$\xrightarrow{R_1-3R_3 \to R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1/2 \\ 0 & 4 & 0 & 0 & 1 & 5/6 \\ 0 & 0 & 1 & 0 & 0 & 1/6 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1/2 \\ 0 & 1 & 0 & 0 & 1/4 & 5/24 \\ 0 & 0 & 1 & 0 & 0 & 1/6 \end{array}\right]$$

$$\xrightarrow{R_1+2R_2 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1/2 & -1/12 \\ 0 & 1 & 0 & 0 & 1/4 & 5/24 \\ 0 & 0 & 1 & 0 & 0 & 1/6 \end{array}\right]$$

So the inverse is

$$\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1/4 & 5/24 \\ 0 & 0 & 1/6 \end{bmatrix}$$

2. (a) Find the inverse of the following matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 0 & 6 & 7 \end{bmatrix}$$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 5 & 0 & 1 & 0 \\ 0 & 6 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5/2 & 0 & 1/2 & 0 \\ 0 & 6 & 7 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - 6R_2 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5/2 & 0 & 1/2 & 0 \\ 0 & 0 & -8 & 0 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{8}R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 3/8 & -1/8 \end{array}\right]$$

$$\xrightarrow{R_2 - \frac{5}{2}R_3 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -7/16 & 5/16 \\ 0 & 0 & 1 & 0 & 3/8 & -1/8 \end{array}\right]$$

$$\xrightarrow{R_1 - 3R_3 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & -9/8 & 3/8 \\ 0 & 1 & 0 & 0 & -7/16 & 5/16 \\ 0 & 0 & 1 & 0 & 3/8 & -1/8 \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_2 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1/4 & -1/4 \\ 0 & 1 & 0 & 0 & -7/16 & 5/16 \\ 0 & 0 & 1 & 0 & 3/8 & -1/8 \end{array}\right]$$

So the inverse is

$$\begin{bmatrix} 1 & -1/4 & -1/4 \\ 0 & -7/16 & 5/16 \\ 0 & 3/8 & -1/8 \end{bmatrix}$$

(b) Let A be the matrix from the previous question and suppose that $BC = A$ where B and C are both 3×3 matrices and B is as shown below. Find C^{-1}.

$$B = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

Solution: It is possible to solve this problem by first finding the inverse of B, multiplying $BC = A$ by B^{-1} on both sides to get $C = B^{-1}A$ and then finding the inverse of $B^{-1}A$. But that's a lot of effort. Instead, lets first observe that if we multiply $BC = A$ on both sides by A^{-1} then we get $A^{-1}(BC) = A^{-1}A = I$. So $(A^{-1}B)C = I$ and by the definition of the matrix inverse, that means that $A^{-1}B$ is the inverse of C. We already found A^{-1} in part (a) and we know what B is. So we can calculate

$$C^{-1} = A^{-1}B = \begin{bmatrix} 1 & -1/4 & -1/4 \\ 0 & -7/16 & 5/16 \\ 0 & 3/8 & -1/8 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 13/4 & 5/4 & 3/4 \\ -5/16 & 3/16 & 5/16 \\ 1/8 & 1/8 & -1/8 \end{bmatrix}$$

3. True or false:

(a) The following vector is an eigenvector of the following matrix.

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 9 \end{bmatrix}$$

Solution: True. Multiplying the matrix by the vector gives

$$\begin{bmatrix} 0 \\ 11 \\ 22 \end{bmatrix}$$

Since this is equal to 11 times the original vector, the original vector is an eigenvector of the matrix (with eigenvalue 11).

(b) If v is an eigenvector of A then v is also an eigenvector of $5A$.

Solution: True. Suppose v is an eigenvector of A with eigenvalue λ. So by definition, $Av = \lambda v$. Thus $(5A)v = 5(Av) = 5(\lambda v) = (5\lambda)v$. So v is an eigenvector of $5A$ with eigenvalue 5λ.

(c) If v is an eigenvector of A and of B then it is also an eigenvector of AB.

Solution: True. Suppose v is an eigenvector of A with eigenvalue λ_1 and also an eigenvalue of B with eigenvalue λ_2. So by definition, $Av = \lambda_1 v$ and $Bv = \lambda_2 v$. Therefore

$$(AB)v = A(Bv) = A(\lambda_2 v) = \lambda_2(Av) = \lambda_2(\lambda_1 v) = (\lambda_2\lambda_1)v.$$

So v is an eigenvector of AB with eigenvalue $\lambda_2\lambda_1$.

(d) If v is an eigenvector of an invertible matrix A then it is also an eigenvector of A^{-1}.

Solution: True. Suppose v is an eigenvector of A with eigenvalue λ. So by definition $Av = \lambda v$. Multiplying both sides by A^{-1} gives us

$$\begin{aligned} A^{-1}(Av) &= A^{-1}(\lambda v) \\ (A^{-1}A)v &= \lambda(A^{-1}v) \\ Iv &= \lambda(A^{-1}v) \\ v &= \lambda(A^{-1}v). \end{aligned}$$

The main idea at this point is just to divide both sides by λ. But to do that, we need to know that λ is not zero. But if λ were zero then by the last line above, v would have to be the all-zeros vector. Since v is an eigenvector, it is by definition not the all-zeros vector. So λ is not zero, and dividing the last equation above by λ gives us

$$\frac{1}{\lambda}v = A^{-1}v$$

and thus v is an eigenvector of A^{-1} with eigenvalue $1/\lambda$.

(e) If v is an eigenvector of A then v is also an eigenvector of A^5.

Solution: True. This is essentially the same as part (c). If v is an eigenvector of A with eigenvalue λ then $A^5 v = \lambda^5 v$ and so v is an eigenvector of A^5 with eigenvalue λ^5.

4. Find the eigenvalues and eigenvectors of the following matrices.

(a)

$$\begin{bmatrix} 2 & 1 \\ -2 & 5 \end{bmatrix}$$

Solution: Eigenvalues: 3, 4
Eigenvectors for eigenvalue 3:

$$\begin{bmatrix} a \\ a \end{bmatrix}$$

where a is any nonzero real number.
Eigenvectors for eigenvalue 3:

$$\begin{bmatrix} b \\ 2b \end{bmatrix}$$

where b is any nonzero real number.

(b)

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Solution: Eigenvalues: 2
Eigenvectors for eigenvalue 2:

$$\begin{bmatrix} a \\ 0 \end{bmatrix}$$

where a is any nonzero real number.

Comment: Note that there is only one eigenvalue even though the matrix is 2×2. This can happen sometimes, though it is not the typical scenario.

(c)

$$\begin{bmatrix} 1/2 & -3/5 \\ 3/4 & 11/10 \end{bmatrix}$$

Solution: Eigenvalues: $\frac{4}{5} + \frac{3}{5}i$ and $\frac{4}{5} - \frac{3}{5}i$.
Eigenvectors for eigenvalue $\frac{4}{5} + \frac{3}{5}i$:

$$\begin{bmatrix} \left(-\frac{2}{5} + \frac{4}{5}i\right) a \\ a \end{bmatrix}$$

where a is any nonzero number.

Eigenvectors for eigenvalue $\frac{4}{5} + \frac{3}{5}i$:

$$\begin{bmatrix} \left(-\frac{2}{5} - \frac{4}{5}i\right) b \\ b \end{bmatrix}$$

where b is any nonzero number.

5. **Challenge Question:** When does a 2×2 matrix whose entries are all integers have an inverse whose entries are all integers? What about for an $n \times n$ matrix?

6. **Challenge Question:** Let $p(x)$ be a degree n polynomial. Can you always find an $n \times n$ matrix whose characteristic polynomial is p?